

REFLECTABLE BASES FOR AFFINE REFLECTION SYSTEMS

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ABSTRACT. The notion of a “root base” together with its geometry plays a crucial role in the theory of finite and affine Lie theory. However, it is known that such a notion does not exist for the recent generalizations of finite and affine root systems such as extended affine root systems and affine reflection systems. In this work, we consider the notion of a “reflectable base” for an affine reflection system R . A reflectable base for R is a minimal subset Π of roots such that the non-isotropic part of the root system can be recovered by reflecting roots of Π relative to the hyperplanes determined by Π . We give a full characterization of reflectable bases for tame irreducible affine reflection systems of reduced types, excluding types $E_{6,7,8}$. As a by-product of our results, we show that if the root system under consideration is locally finite, then any reflectable base is an integral base.

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0. Introduction

Over recent years there has been an increasing amount of investigations on topics related to extended affine root systems, extended affine Lie algebras and their generalizations. However, in comparison with the inspiring models of finite and affine cases (see [Mac], [Hum], [K], [MP]), only a little is known about the geometry of involved root systems (see [Hof2]). In the finite and affine Lie theory, the notion of a “root base” plays a crucial role in the study of not only the corresponding geometry but also the whole theory, whereas it is known that such a notion does not exist in general for the new generalizations. In this work, we introduce the notions of a *reflectable set* and a *reflectable base* for a *tame irreducible affine reflection system*, and we characterize reflectable sets and reflectable bases for tame irreducible affine reflection systems of reduced types, excluding types $E_{6,7,8}$.

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Affine reflection systems are defined axiomatically in a way similar to extended affine root systems. In fact, the axioms are the same as those for extended affine root systems, except that the underlying finite dimensional vector space is replaced with an arbitrary abelian group; see Definition 1.3. The notion of an affine reflection system introduced here is more general than the one defined by O. Loos and E. Neher [LN2], in fact these two definitions are equivalent if the ground abelian group in our definition is torsion free; see Remark 1.16. Affine reflection systems include extended affine root systems [AABGP], locally extended affine root systems [MY], and root systems extended by an abelian group [Y2].

A reflectable set for a tame irreducible affine reflection system is a subset Π of non-isotropic roots such that any non-isotropic root can be obtained by reflecting a root of Π relative to hyperplanes determined by Π . A reflectable base is a minimal reflectable set; see Definition 1.19. One knows that any “root base” for a finite or affine root system is a reflectable base, and it follows from our results in this work that, for the types under consideration, the notion of a reflectable base for a locally finite root system coincides with the one introduced by Y. Yoshii [Y3]. Also it is proved in [MS], [A2] and [LN1, Lemma 5.1] that any extended affine root system or locally finite root system of reduced type possesses a reflectable base. For extended affine root systems of nullity 2, a notion of “root base” is introduced by K. Saito [Sa], and some of its algebraic and geometric features are studied, however this notion of root base is not in general a reflectable base as it might fail to have the minimality condition. In the literature, one finds several other related terms such as “generalized bases” [NS], “integral bases” [LN1], “grid bases” [N2], etc., each resembles, in some aspects, the usual notion of a root base for finite and affine cases. It is shown in [N2] that any grid base for a 3-graded root system is a reflectable base.

Reflectable bases have been appeared, though not necessarily with such a name, in different contexts such as the description of root systems, presentations of Weyl groups and presentations of Lie algebras [Sa], [SaT], [AS1], [AS2], [AS3], [AS4], [Hof1] and [SaY]. In the level of Lie algebras and Weyl groups, an essential part of generators for a given presentation is obtained from reflectable bases. The Weyl group acts naturally on the class of reflectable bases of an affine reflection system. In the finite and affine cases, some orbits of this action play a very important role, namely in the finite case the class of root bases forms exactly one orbit of this action and for the affine case, this class is the union of exactly two orbits; see [K]. From this point of view, the study of orbits of reflectable bases seems to be an interesting subject of research, this has been one of our motivations for the study of such objects. It is also known that the Weyl group of an affine reflection system is not in general a Coxeter group, for example no extended affine Weyl group of nullity greater than or equal 2 is a Coxeter group; see [Hof1]. Nevertheless, non-Coxeter Weyl groups are revealed to have interesting geometric structures as well [Hof2]. This has been another motivation for us to do this work.

In this work, we give a full characterization of reflectable bases for tame irreducible affine reflection systems of the types under consideration, namely types A , B , C , D , F

and G ; such a characterization has not been known even for the finite and affine cases. To establish such a characterization for types $E_{6,7,8}$, one needs further investigations which require an independent study.

The paper is arranged as follows. In Section 1, we introduce axiomatically the notion of an affine reflection system R in the same framework of an extended affine root system (see Definition 1.3) and describe the structure of R in the same lines of [AABGP] in terms of an involved locally finite root system and some (pointed) reflection subspaces of the ground abelian group; see Theorem 1.13. The relation between affine reflection systems and certain other generalizations of finite and affine root systems is clarified in Remark 1.16. The notions of reflectable sets, reflectable bases and integral bases are defined for affine reflection systems, and some preliminary results are obtained for them. Some special subsets of roots, which play an important role in our characterization of reflectable sets and reflectable bases, are introduced, including *coset spanning sets*, *coset bases* and *strong coset spanning sets*.

In Section 2, we characterize reflectable bases, and reflectable sets for irreducible locally finite root systems of types under consideration. For simply laced cases, a subset of nonzero roots is a reflectable base if and only if it is a minimal set of generators for the ground abelian group (see Proposition 2.5) if and only if its image, under the canonical map, is a basis for the \mathbb{Z}_2 -vector space $\langle \dot{R} \rangle / 2\langle \dot{R} \rangle$, where $\langle \cdot \rangle$ denotes the \mathbb{Z} -span. This is what we call a coset basis; see Propositions 2.6 and 2.9. For non-simply laced types, the reflectable sets and bases are characterized in terms of coset spanning sets and coset bases of short and long roots in some appropriate vector spaces. For example, according to our characterization, a subset of an irreducible locally finite root system of type B , of rank ≥ 3 , is a reflectable base if and only if it is the union of a short root and a coset basis of long roots in $2\langle \dot{R}_{sh} \rangle$; see Proposition 2.4. The section contains also an expected but totally non-trivial fact that any reflectable set contains a reflectable base; see Corollary 2.14 and Propositions 2.6 and 2.9. As a by-product of our results in this section we see that any reflectable base for an irreducible locally finite root system, of types under consideration, is an integral base. In other words, any reflectable base is a basis for the underlying free abelian group. It is also shown that, for non-simply laced types, the cardinality $|\dot{\Pi}|$ of a reflectable base $\dot{\Pi}$ can be characterized in terms of dimensions of some vector spaces over Galois fields, namely $|\dot{\Pi}| = \dim(\langle \dot{R}_{sh} \rangle / \langle \dot{R}_{lg} \rangle) + \dim(\langle \dot{R}_{lg} \rangle / \langle \rho \dot{R}_{sh} \rangle)$, where here ρ stands for the ratio of the long square root length to the short square root length in \dot{R} ; see Remark 2.15.

In Section 3, using our results for locally finite root systems, we give a full characterization of reflectable sets and reflectable bases for tame irreducible affine reflection systems, of the types under consideration, in terms of strong coset spanning sets and minimal strong coset spanning sets. To see a flavor of our results, let R be an tame irreducible affine reflection system with the sets of short roots R_{sh} and long roots R_{lg} . Then for type A_1 , a subset Π of R is a reflectable set (resp. a reflectable base) for R if and only if Π is a strong coset spanning set (resp. a minimal strong coset spanning set) for R^\times in $2\langle R \rangle$; see Theorem 3.1. For type B_2 , a subset Π of R satisfying $\langle \Pi \rangle = \langle R \rangle$ is a reflectable set (resp. reflectable base) for R if and only if $\Pi_{sh} := \Pi \cap R_{sh}$ is a strong

coset spanning set (resp. a minimal strong coset spanning set) for R_{sh} in $\langle R_{lg} \rangle$ and $\Pi_{lg} := \Pi \cap R_{lg}$ is a strong coset spanning set (resp. a minimal strong coset spanning set) for R_{lg} in $2\langle R_{sh} \rangle$; see Theorem 3.14.

We hope our characterization of reflectable sets and reflectable bases offers a new perspective to the study of geometry of affine reflection systems. The authors dedicate this work to the memory of V. Shahsanaei who passed away in a fatal car accident at the very early stage of this project, he was supposed to be one of the authors.

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1. Affine reflection systems

Throughout this work, all vector spaces are considered over \mathbb{Q} unless otherwise mentioned. For a vector space \mathcal{V} , by \mathcal{V}^* , we mean the dual space of \mathcal{V} . Also for a subset X of a group A , we denote by $\langle X \rangle$, the subgroup of A generated by X and by $\text{Aut}(A)$, we mean the group of automorphisms of A . In this work for a set K , $|K|$ denotes the cardinality of K and we use the notation \uplus , for disjoint union. We recall from [L] that a *pointed reflection subspace* of an additive abelian group A is a subset X of A satisfying one of the following equivalent conditions:

- $0 \in X$ and $X - 2X \subseteq X$,
- $0 \in X$ and $2X - X \subseteq X$,
- $2\langle X \rangle \subseteq X$ and $2\langle X \rangle - X \subseteq X$,
- X is a union of cosets of $2\langle X \rangle$ in X , including the trivial coset $2\langle X \rangle$.

Also a *symmetric reflection subspace* of A is a subset X of A satisfying one of the following equivalent conditions:

- $X - 2X \subseteq X$,
- $X = -X$ and $2X + X \subseteq X$,
- $X = -X$ and $2X - X \subseteq X$.

Let A be an abelian group, by a *symmetric form* on A , we mean a symmetric bi-homomorphism $(\cdot, \cdot) : A \times A \rightarrow \mathbb{Q}$. This means that (\cdot, \cdot) is a group homomorphism on each component and is symmetric, namely $(\alpha, \alpha') = (\alpha', \alpha)$ and

$$(\alpha + \beta, \alpha' + \beta') = (\alpha, \alpha') + (\alpha, \beta') + (\beta, \alpha') + (\beta, \beta')$$

for all $\alpha, \alpha', \beta, \beta' \in A$. We set $A^0 := \{\alpha \in A \mid (\alpha, A) = \{0\}\}$, and we call it the *radical* of (\cdot, \cdot) . We also set $A^\times := A \setminus A^0$, $\bar{A} := A/A^0$ and take $\bar{\cdot} : A \rightarrow \bar{A}$ to be the canonical epimorphism. For a subset Y of A , we denote by \bar{Y} , the image of Y under the map $\bar{\cdot}$. We note that \bar{A} is a torsion free group. The form (\cdot, \cdot) is called positive semidefinite (resp. positive definite) if $(\alpha, \alpha) \geq 0$ (resp. $(\alpha, \alpha) > 0$) for all $\alpha \in A \setminus \{0\}$, in this case one can see that

$$A^0 = \{\alpha \in A \mid (\alpha, \alpha) = 0\}.$$

For a subset B of A , we set $B^\times := B \setminus A^0$ and $B^0 := B \cap A^0$. For $\alpha, \beta \in A$, if $(\alpha, \alpha) \neq 0$, we set $(\beta, \alpha^\vee) := 2(\beta, \alpha)/(\alpha, \alpha)$ and if $(\alpha, \alpha) = 0$, we set $(\beta, \alpha^\vee) := 0$. A subset X

of A is called *connected* if it cannot be written as a disjoint union of two nonempty orthogonal subsets. The form (\cdot, \cdot) induces a unique form on \bar{A} by

$$(\bar{\alpha}, \bar{\beta}) := (\alpha, \beta) \quad \text{for } \alpha, \beta \in A.$$

This form is positive definite on \bar{A} .

Next suppose that R is a subset of A satisfying $\langle R \rangle = A$ and $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R^\times$. For $\alpha \in R^\times$, we take $w_\alpha \in \text{Aut}(A)$ to be defined by $w_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$, $\beta \in A$, and call it the *reflection based on α* . We define the *Weyl group* \mathcal{W} of R to be the subgroup of $\text{Aut}(A)$ generated by w_α , $\alpha \in R^\times$. In a similar way, one defines $w_{\bar{\alpha}} \in \text{Aut}(\bar{A})$ and $\bar{\mathcal{W}}$, the subgroup of $\text{Aut}(\bar{A})$ generated by $w_{\bar{\alpha}}$, $\bar{\alpha} \in \bar{R} \setminus \{0\}$. One can see that

$$sw_\alpha s^{-1} = w_{s\alpha}; \quad s \in \mathcal{W} \quad \text{and} \quad \alpha \in R^\times. \quad (1.1)$$

For a subset \mathcal{P} of R^\times , we set

$$\mathcal{W}_{\mathcal{P}} := \langle w_\alpha \mid \alpha \in \mathcal{P} \rangle \quad \text{and} \quad \mathcal{W}_{\mathcal{P}}\mathcal{P} := \{w(\alpha) \mid w \in \mathcal{W}_{\mathcal{P}}, \alpha \in \mathcal{P}\}. \quad (1.2)$$

Definition 1.3. Let A be an abelian group equipped with a nontrivial symmetric positive semidefinite form (\cdot, \cdot) . Let R be a subset of A . The triple $(A, (\cdot, \cdot), R)$, or R if there is no confusion, is called an *affine reflection system* if it satisfies the following 3 axioms:

- (R1) $R = -R$,
- (R2) $\langle R \rangle = A$,
- (R3) for $\alpha \in R^\times$ and $\beta \in R$, there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that

$$(\beta + \mathbb{Z}\alpha) \cap R = \{\beta - d\alpha, \dots, \beta + u\alpha\} \quad \text{and} \quad d - u = (\beta, \alpha^\vee).$$

Each element of R is called a *root*. Elements of R^\times (resp. R^0) are called *non-isotropic roots* (resp. *isotropic roots*). The affine reflection system R is called *irreducible* if it satisfies

- (R4) R^\times is connected.

Moreover, R is called *tame* if

- (R5) $R^0 \subseteq R^\times - R^\times$ (elements of R^0 are non-isolated).

Finally R is called *reduced* if it satisfies

- (R6) $\alpha \in R^\times \Rightarrow 2\alpha \notin R$.

An affine reflection system $(A, (\cdot, \cdot), R)$ is called a *locally finite root system* if $A^0 = \{0\}$.

Remark 1.4. If R is a finite root system in a Euclidean space E , then R is a locally finite root system in $\langle R \rangle$. Therefore we consider a finite root system as a locally finite root system.

Let $(A, (\cdot, \cdot), R)$ be an affine reflection system. Here we describe the structure of R in the same lines of [AABGP] for extended affine root systems. Since the form is nontrivial, we have $A \neq A^0$. Now this together with (R1)-(R3), implies that

$$0 \in R. \quad (1.5)$$

Lemma 1.6. *If $(A, (\cdot, \cdot), R)$ is an affine reflection system, then the induced form on $\mathcal{V} := \mathbb{Q} \otimes_{\mathbb{Z}} \bar{A}$ is non-degenerate. In particular, $1 \otimes \bar{R}$ is locally finite in \mathcal{V} , that is, any finite dimensional vector subspace of \mathcal{V} intersects $1 \otimes \bar{R}$ in a finite set.*

Proof. Denote the induced form on \mathcal{V} by (\cdot, \cdot) again. We recall that this form satisfies

$$(r \otimes \bar{a}, s \otimes \bar{b}) = rs(\bar{a}, \bar{b}); \quad r, s \in \mathbb{Q}, \quad a, b \in A.$$

Suppose that n is a positive integer, $\{p_i/q_i \mid 1 \leq i \leq n\} \subseteq \mathbb{Q}$, $\{a_i \mid 1 \leq i \leq n\} \subseteq A$ and $\sum_{i=1}^n (p_i/q_i) \otimes \bar{a}_i$ is an element of the radical of the form on \mathcal{V} . Take $q := \prod_{j=1}^n q_j$ and $p'_i := p_i \prod_{j \neq i}^n q_j$, $1 \leq i \leq n$, then

$$(1/q) \sum_{i=1}^n 1 \otimes p'_i \bar{a}_i = (1/q) \sum_{i=1}^n p'_i \otimes \bar{a}_i = \sum_{i=1}^n (p'_i/q) \otimes \bar{a}_i = \sum_{i=1}^n (p_i/q_i) \otimes \bar{a}_i$$

is an element of the radical of the form which in turn implies that $\sum_{i=1}^n 1 \otimes p'_i \bar{a}_i$ is an element of the radical. Therefore for all $a \in A$, $(\sum_{i=1}^n p'_i \bar{a}_i, \bar{a}) = (1 \otimes \sum_{i=1}^n p'_i \bar{a}_i, 1 \otimes \bar{a}) = 0$. This means that $\sum_{i=1}^n p'_i \bar{a}_i$ is an element of the radical of the form on \bar{A} . Now as the form on \bar{A} is non-degenerate, we are done. For the last assertion, using the same argument as in [AABGP, Lem. I.2.6], one can see that

$$-4 \leq 2(\beta, \alpha)/(\alpha, \alpha) \leq 4; \quad \alpha, \beta \in R^\times.$$

This together with the first part of the proof and the same argument as in [MY, Pro. 3.7] completes the proof. \square

Proposition 1.7. *Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root system, then $\tilde{R} := 1 \otimes R \subseteq \mathcal{V} := \mathbb{Q} \otimes_{\mathbb{Z}} A$ satisfies the following*

- (a) $0 \in \tilde{R}$, $\text{span}_{\mathbb{Q}} \tilde{R} = \mathcal{V}$ and \tilde{R} is locally finite,
- (b) for every $\tilde{\alpha} \in \tilde{R} \setminus \{0\}$, there exists $\check{\alpha} \in \mathcal{V}^*$ such that $\check{\alpha}(\tilde{\alpha}) = 2$ and \tilde{R} is invariant under the reflection $s_{\check{\alpha}} : \mathcal{V} \rightarrow \mathcal{V}$ mapping $v \in \mathcal{V}$ to $v - \check{\alpha}(v)\tilde{\alpha}$,
- (c) $\check{\alpha}(\tilde{\beta}) \in \mathbb{Z}$, for $\tilde{\alpha}, \tilde{\beta} \in \tilde{R} \setminus \{0\}$.

Conversely, if \mathcal{V} is a vector space and $\tilde{R} \subseteq \mathcal{V}$ satisfies (a)-(c) above, then \mathcal{V} is equipped with a symmetric positive definite bilinear form (\cdot, \cdot) invariant under $s_{\check{\alpha}}$, $\tilde{\alpha} \in \tilde{R} \setminus \{0\}$. Moreover setting $A := \langle \tilde{R} \rangle$, we have that $(A, (\cdot, \cdot)|_{A \times A}, \tilde{R})$ is a locally finite root system.

Proof. Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root system. Since $\langle R \rangle = A$, we get that

$$\text{span}_{\mathbb{Q}} \tilde{R} = \mathcal{V}.$$

Next using Lemma 1.6, we get that

$$\tilde{R} \text{ is locally finite in } \mathcal{V}.$$

Now using (R3), we get that for $\alpha, \beta \in R^\times$, $2(1 \otimes \beta, 1 \otimes \alpha)/(1 \otimes \alpha, 1 \otimes \alpha) \in \mathbb{Z}$ and that the linear map $\text{id} \otimes w_\alpha : \mathcal{V} \rightarrow \mathcal{V}$ preserves \tilde{R} . Now if we set $\tilde{\alpha} := 1 \otimes \alpha$ and define $\check{\alpha} \in \mathcal{V}^*$ by $v \mapsto 2(v, \tilde{\alpha})/(\tilde{\alpha}, \tilde{\alpha})$, then one gets that (a)-(c) are fulfilled. Conversely, suppose that \mathcal{V} is a vector space and $\tilde{R} \subseteq \mathcal{V}$ satisfies (a)-(c), then by [LN1, Thm. 4.2],

there is a positive definite symmetric bilinear form on \mathcal{V} with desired property. Now using [LN1, §4], the proof of Proposition 3.15 of [LN1] and general facts on finite root systems, we are done. \square

Proposition 1.8. *Let A be an abelian group and R be a subset of A . Then there is a positive definite symmetric form (\cdot, \cdot) on A such that the triple $(A, (\cdot, \cdot), R)$ is a locally finite root system if and only if A is a free abelian group of rank $\dim(\mathcal{V})$ where $\mathcal{V} := \mathbb{Q} \otimes_{\mathbb{Z}} A$ and R satisfies*

- (a) $0 \in R$, $\langle R \rangle = A$ and R is locally finite in the sense that any subgroup of A of finite rank intersects R in a finite subset,
- (b) for every $\alpha \in R^\times$, there exists $\check{\alpha} \in \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q})$ such that $\check{\alpha}(\alpha) = 2$ and R is invariant under the reflection $s_\alpha : A \longrightarrow A$ mapping $a \in A$ to $a - \check{\alpha}(a)\alpha$,
- (c) $\check{\alpha}(\beta) \in \mathbb{Z}$, for $\alpha, \beta \in R^\times$.

Proof. Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root system, then Proposition 1.7 together with [LN2, Lem. 5.1] implies that $1 \otimes R$ possesses a subset B satisfying the followings:

- 1) B is a basis for \mathcal{V} ,
- 2) each element of $1 \otimes R$ can be written as a \mathbb{Z} -linear combination of elements of B .

This shows that $\langle 1 \otimes R \rangle = \langle B \rangle$ is a free abelian group. This in turn together with the fact that A is torsion free and $B \subseteq 1 \otimes R$ implies that $A = \langle R \rangle$ is a free abelian group of desired rank. Next suppose that X is a subgroup of A of finite rank, then there is a finite subset \mathcal{C} of B such that $1 \otimes X \subseteq \langle \mathcal{C} \rangle$. But $(1 \otimes R) \cap (1 \otimes X) \subseteq (1 \otimes R) \cap \langle \mathcal{C} \rangle$ and so $(1 \otimes R) \cap (1 \otimes X) \subseteq (1 \otimes R) \cap \text{span}_{\mathbb{Q}}(\mathcal{C})$. Now since $\mathcal{U} := \text{span}_{\mathbb{Q}}(\mathcal{C})$ is a finite dimensional subspace of \mathcal{V} and by Proposition 1.7, $1 \otimes R$ is locally finite, we get that $(1 \otimes R) \cap \mathcal{U}$ is finite. Therefore $(1 \otimes R) \cap (1 \otimes X)$ and so $R \cap X$ is finite. In other words, (a) is satisfied. Next to see that (b) and (c) are fulfilled, for $\alpha \in R^\times$ define

$$\check{\alpha} \in \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}); \quad a \mapsto 2(\alpha, a)/(\alpha, \alpha)$$

an use (R3).

Conversely, suppose that R is a subset of a free abelian group A for which (a)-(c) are satisfied. We first show that $1 \otimes R$ is locally finite in $\mathcal{V} = \mathbb{Q} \otimes_{\mathbb{Z}} A$. Take $\{a_i \mid i \in I\}$ to be a basis for the free abelian group A and suppose that \mathcal{U} is a finite dimensional subspace of \mathcal{V} , then there is a finite subset J of I such that $\mathcal{U} \subseteq \text{span}_{\mathbb{Q}}\{1 \otimes a_j \mid j \in J\}$. Now it follows from the facts that $\{1 \otimes a_i \mid i \in I\}$ and $\{a_i \mid i \in I\}$ are bases for the vector space \mathcal{V} and the free abelian group A respectively, that if $1 \otimes \alpha \in (1 \otimes R) \cap \mathcal{U}$, then $\alpha \in B \cap R$ in which B is the subgroup of A generated by $\{a_j \mid j \in J\}$. Then as R is locally finite in A , we get that $B \cap R$ is finite and so $(1 \otimes R) \cap \mathcal{U}$ is finite, i.e., $1 \otimes R$ is locally finite in \mathcal{V} . Now for $\tilde{\alpha} := 1 \otimes \alpha$, $\alpha \in R^\times$, define

$$\check{\alpha} : \mathcal{V} \longrightarrow \mathbb{Q}; \quad q \otimes a \mapsto q\check{\alpha}(a); \quad q \in \mathbb{Q}, a \in A.$$

One can easily check that conditions (a)-(c) in Proposition 1.7 are satisfied for $\tilde{R} := 1 \otimes R \subseteq \mathcal{V}$. Therefore \mathcal{V} is equipped with a positive definite symmetric bilinear form (\cdot, \cdot) such that $(1 \otimes A, (\cdot, \cdot)_{|(1 \otimes A) \times (1 \otimes A)}, 1 \otimes R)$ is a locally finite root system. Now we are done as the map from A to \mathcal{V} mapping $a \in A$ to $1 \otimes a$ is an embedding. \square

Corollary 1.9. *If $(A, (\cdot, \cdot), R)$ is an affine reflection system, then $(\bar{R}, (\cdot, \cdot), \bar{A})$ is a locally finite root system. In particular, if R is irreducible, the induced form on $\mathcal{V} := \mathbb{Q} \otimes_{\mathbb{Z}} \bar{A}$ is positive definite.*

Proof. For the first statement, using Lemma 1.6 and a minor modification of the proof of Proposition 1.8, we are done. The second assertion follows from Proposition 1.7, [LN1, Thm. 4.2] and the facts that the form (\cdot, \cdot) on \bar{A} is positive definite and that this form is invariant under $w_{\bar{\alpha}}, \alpha \in R^\times$. \square

Now suppose that \dot{R} is a locally finite root system in an abelian group \dot{A} . Using Proposition 1.8, one gets that \dot{A} is a free abelian group. We define the *rank* of \dot{R} to be the rank of \dot{A} . A subset \dot{S} of \dot{R} is said to be a *subsystem* of \dot{R} if it contains zero and $\dot{w}_{\dot{\alpha}}(\dot{\beta}) \in \dot{S}$ for $\dot{\alpha}, \dot{\beta} \in \dot{S} \setminus \{0\}$. Two locally finite root systems (\dot{R}, \dot{A}) and (\dot{S}, \dot{B}) are said to be isomorphic if there is a group isomorphism $f : \dot{A} \rightarrow \dot{B}$ such that $f(\dot{R}) = \dot{S}$.

Suppose that I is a nonempty index set and $\dot{A} := \oplus_{i \in I} \mathbb{Z}\epsilon_i$ is the free abelian group on the set I . Define the form

$$\begin{aligned} (\cdot, \cdot) : \dot{A} \times \dot{A} &\rightarrow \mathbb{Q}, \\ (\epsilon_i, \epsilon_j) &:= \delta_{i,j}, \text{ for } i, j \in I. \end{aligned}$$

This is a positive definite symmetric form on \dot{A} . Next define

$$\begin{aligned} \dot{A}_I &:= \{\epsilon_i - \epsilon_j \mid i, j \in I\}, \\ \dot{D}_I &:= \dot{A}_I \cup \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in I, i \neq j\}, \\ \dot{B}_I &:= \dot{D}_I \cup \{\pm\epsilon_i \mid i \in I\}, \\ \dot{C}_I &:= \dot{D}_I \cup \{\pm 2\epsilon_i \mid i \in I\}, \\ \dot{BC}_I &:= \dot{B}_I \cup \dot{C}_I. \end{aligned} \tag{1.10}$$

One can see that these are irreducible locally finite root systems in their \mathbb{Z} -span's. Moreover, every irreducible locally finite root system of infinite rank is isomorphic to one of these root systems; see Proposition 1.7 and [LN1, §4.14 and §8]. Now for an irreducible locally finite root system \dot{R} , define

$$\begin{aligned} \dot{R}_{sh} &:= \{\dot{\alpha} \in \dot{R}^\times \mid (\dot{\alpha}, \dot{\alpha}) \leq (\dot{\beta}, \dot{\beta}); \text{ for all } \dot{\beta} \in \dot{R}\}, \\ \dot{R}_{ex} &:= \dot{R} \cap 2\dot{R}_{sh}, \\ \dot{R}_{lg} &:= \dot{R}^\times \setminus (\dot{R}_{sh} \cup \dot{R}_{ex}). \end{aligned}$$

The elements of \dot{R}_{sh} (resp. $\dot{R}_{lg}, \dot{R}_{ex}$) are called *short roots* (resp. *long roots*, *extra-long roots*) of \dot{R} . Using Proposition 1.7 and [LN1, Cor. 5.6], one gets that each two nonzero roots of \dot{R} of the same length are conjugate under the Weyl group of \dot{R} . In the following, for not overusing the notations, irreducible finite root systems of types $A_{\ell-1}, B_\ell, C_\ell, D_\ell$ and BC_ℓ will be denoted by $\dot{A}_I, \dot{B}_I, \dot{C}_I, \dot{D}_I$ and \dot{BC}_I respectively in which I is an index set of cardinality ℓ . We also refer to locally finite root systems of types $\dot{A}_I, \dot{D}_I, E_6, E_7$ and E_8 as simply laced types.

Definition 1.11. Considering Corollary 1.9, for an affine reflection system R , we call the type and the rank of \bar{R} to be the *type* and the *rank* of R , respectively.

The following proposition is a generalization of Proposition 5.9 of [Hof1] to affine reflection systems.

Proposition 1.12. *Suppose that A is an additive abelian group equipped with a positive semidefinite symmetric form (\cdot, \cdot) and B is a subset of A^\times satisfying*

- $\mathcal{W}_B B \subseteq B$,
- \bar{B} is an irreducible locally finite root system in $\text{span}_{\mathbb{Z}} \bar{B}$.

Then $R := B \cup ((B - B) \cap A^0)$ is a tame irreducible affine reflection system in $\text{span}_{\mathbb{Z}}(B)$.

Proof. It is easy to see that (R1), (R2), (R4) and (R5) hold, so we just need to check that (R3) is satisfied. Let $\alpha \in R^\times$ and $\beta \in R$. We will look at three different cases.

Case I: $\bar{\alpha}$ and $\bar{\beta}$ are linearly independent over \mathbb{Z} , that is the subgroup of \bar{A} generated by $\bar{\alpha}, \bar{\beta}$ is of rank 2 : Set $R_{\alpha, \beta} := R \cap (\mathbb{Z}\alpha \oplus \mathbb{Z}\beta)$. Since the form on \bar{A} is positive definite and $\bar{\alpha}, \bar{\beta}$ are \mathbb{Z} -linearly independent, it is easy to see that the restriction of the form to $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ is positive definite. Now since $(\beta', \alpha'^\vee) \in \mathbb{Z}$ for all $\alpha', \beta' \in R_{\alpha, \beta}$, it follows using the same argument as in [AABGP, Lem. I.2.6] that $R_{\alpha, \beta}$ is finite. Therefore it is a finite root system in $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ and so (R3) holds in this case.

Case II: $\beta \in R^0$: If $n \in \mathbb{Z}$ and $\beta + n\alpha \in R$, then since $\mathcal{W}_B B \subseteq B$, $\beta - n\alpha = w_\alpha(\beta + n\alpha) \in B$. So the set of integer numbers n for which $\bar{\beta} + n\bar{\alpha} = n\bar{\alpha} \in \bar{R}$ is $\{0\}$, $\{0, \pm 1\}$ or $\{0, \pm 1, \pm 2\}$. But if $\beta \pm 2\alpha \in R$, again using the fact that $\mathcal{W}_B B \subseteq B$, one gets that $\beta \pm \alpha = w_{\beta \pm 2\alpha}(-\alpha) \in B$. So $\{\beta + n\alpha \mid n \in \mathbb{Z}\} \cap R$ is equal $\{\beta\}$, $\{\beta - \alpha, \beta, \beta + \alpha\}$ or $\{\beta - 2\alpha, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha\}$. Therefore in each case (R3) holds.

Case III: $\bar{\alpha}$ and $\bar{\beta}$ are nonzero and linearly dependent over \mathbb{Z} : Since $\bar{R} = \bar{B}$ is a locally finite root system, we only need to consider three cases $\bar{\alpha} = \bar{\beta}$, $\bar{\alpha} = 2\bar{\beta}$ or $\bar{\beta} = 2\bar{\alpha}$. We show that in each case the α -string through β is of the desired form. If $\bar{\alpha} = 2\bar{\beta}$, then $w_\alpha(\beta) = \beta - \alpha$ and so the α -string through β is nothing but $\{\beta - \alpha, \beta\}$ as for $r \in \mathbb{Z}^{\leq -2} \cup \mathbb{Z}^{\geq 1}$, $\overline{\beta + r\alpha} = k\bar{\beta}$ for some integer k with $|k| \geq 3$ which is not an element of \bar{R} . For two other cases, we note that, by Case II, we are done if we show that the mentioned string intersects R^0 . We carry out this as follows:

- (i) $\bar{\alpha} = \bar{\beta}$: In this case we have $\alpha - \beta \in (B - B) \cap A^0 = R^0$.
- (ii) $\bar{\beta} = 2\bar{\alpha}$: In this case the α -string through β is a subset of $\{\beta - 4\alpha, \beta - 3\alpha, \beta - 2\alpha, \beta - \alpha, \beta\}$. The following three cases can happen:
 - $\beta - 2\alpha \in R$: Since $\beta - 2\alpha \in A^0$, $\beta - 2\alpha \in R^0$ and so we are done.
 - $\gamma := \beta - 3\alpha \in R$: In this case $\bar{\gamma} = -\bar{\alpha}$ and so as in Case III(i), we have $\beta - 2\alpha = \gamma + \alpha \in R^0$.
 - $\eta := \beta - 4\alpha \in R$: In this case, $\bar{\eta} = -2\bar{\alpha}$. This implies that $\eta \in B$ and so $\gamma = \beta - 3\alpha = w_\eta \alpha \in \mathcal{W}_B B \subseteq B$. Now using the previous case, we are done.

This completes the proof. \square

Theorem 1.13. *Suppose that $(\dot{A}, (\cdot, \cdot), \dot{R})$ is an irreducible locally finite root system and G is an abelian group. If $\dot{R}_{lg} \neq \emptyset$, set*

$$\rho := (\dot{\beta}, \dot{\beta})/(\dot{\alpha}, \dot{\alpha}), \quad (\dot{\alpha} \in \dot{R}_{sh}, \dot{\beta} \in \dot{R}_{lg}). \quad (1.14)$$

Let S, L, E be subsets of G satisfying the conditions (\star) and (i) – (iv) below:

$$(\star) \quad \begin{array}{l} S, L \text{ (if } \dot{R}_{lg} \neq \emptyset \text{) are pointed reflection subspaces of } G \text{ and} \\ E \text{ (if } \dot{R}_{ex} \neq \emptyset \text{) is a symmetric reflection subspace of } G, \end{array}$$

$$(i) \quad \langle S \rangle = G,$$

$$(ii) \quad S + L \subseteq S, \quad L + \rho S \subseteq L \quad (\dot{R}_{lg} \neq \emptyset),$$

$$S + E \subseteq S, \quad E + 4S \subseteq E \quad (\dot{R} = BC_1),$$

$$L + E \subseteq L, \quad E + 2L \subseteq E \quad (\dot{R} = BC_I, \quad |I| \geq 2),$$

$$(iii) \quad \text{if } \dot{R} \text{ is not of types } A_1, B_I \text{ or } BC_I, \text{ then } S = G,$$

$$(iv) \quad \text{if } \dot{R} \text{ is of types } B_I, F_4, G_2 \text{ or } BC_I \text{ with } |I| \geq 3, \text{ then } L \text{ is a subgroup of } G.$$

Extend (\cdot, \cdot) to a form on $A := \dot{A} \oplus G$ such that $(A, G) = (G, A) = \{0\}$, and set

$$(\star\star) \quad R := (S + S) \cup \underbrace{(\dot{R}_{sh} + S)}_{R_{sh}} \cup \underbrace{(\dot{R}_{lg} + L)}_{R_{lg}} \cup \underbrace{(\dot{R}_{ex} + E)}_{R_{ex}} \subseteq A$$

where if \dot{R}_{lg} or \dot{R}_{ex} is empty, the corresponding parts vanish. Then $(R, (\cdot, \cdot), A)$ is a tame irreducible affine reflection system. Conversely, suppose that $(R, (\cdot, \cdot), A)$ is a tame irreducible affine reflection system, then there is an irreducible locally finite root system \dot{R} and subsets $S, L, E \subseteq G := A^0$ as in (\star) satisfying (i)–(iv) such that R has an expression as $(\star\star)$.

Proof. Using Proposition 1.12, we get the first implication. Conversely suppose that $(A, (\cdot, \cdot), R)$ is a tame irreducible affine reflection system. Since \bar{A} is torsion free, one can identify $\bar{\alpha} \in \bar{A}$ with $1 \otimes \bar{\alpha} \in \mathbb{Q} \otimes_{\mathbb{Z}} \bar{A}$. So from now on we consider \bar{A} as a subset of the \mathbb{Q} -vector space $\mathcal{V} = \mathbb{Q} \otimes_{\mathbb{Z}} \bar{A}$. Now by Corollary 1.9, \bar{R} is a locally finite root system in \mathcal{V} . Considering the proof of Proposition 1.8 and using Proposition 1.7 and [LN2, Lem. 5.1], one gets that \bar{A} is a free abelian group and there is a basis $\bar{\Pi} \subseteq \bar{R}$ for \bar{A} satisfying $\mathcal{W}_{\bar{\Pi}} \bar{\Pi} = \bar{R}^\times$ (sets of this form will be called integral reflectable bases later on). We fix a pre-image $\dot{\Pi} \subseteq R$ of $\bar{\Pi}$ under the projection map $\bar{\cdot}$ and we set $\dot{A} := \langle \dot{\Pi} \rangle$. Since $\bar{\Pi}$ is a \mathbb{Z} -basis of \bar{A} , it follows that $A = \dot{A} \oplus A^0$. We now set

$$\dot{R} := (R + A^0) \cap \dot{A}.$$

Then an argument analogous to [AABGP, §2.2] shows that \dot{R} is a locally finite root system in \dot{A} isomorphic to \bar{R} . Again using exactly the same arguments as in [AABGP, §2.2], one sees that if for $\dot{\alpha} \in \dot{R}^\times$, we set $S_{\dot{\alpha}} := (\dot{\alpha} + A^0) \cap R$, then $S_{\dot{\alpha}} = S_{\dot{\beta}}$ if $\dot{\alpha}$ and $\dot{\beta}$ are of the same length and that setting

$$\begin{aligned} S &:= S_{\dot{\alpha}}; \quad (\dot{\alpha} \in \dot{R}_{sh}), \\ L &:= S_{\dot{\alpha}}; \quad (\dot{\alpha} \in \dot{R}_{lg} \text{ if } \dot{R}_{lg} \text{ is nonempty}), \\ E &:= S_{\dot{\alpha}}; \quad (\dot{\alpha} \in \dot{R}_{ex} \text{ if } \dot{R}_{ex} \text{ is nonempty}), \end{aligned}$$

we have (S, L, E) is as in (\star) and R has an expression as in $(\star\star)$. \square

Proposition 1.15. *Suppose that $(A, (\cdot, \cdot), R)$ is an affine reflection system and \mathcal{C} is a connected subset of R^\times . Then*

- (i) $S := \langle \mathcal{C} \rangle \cap R$ is an irreducible affine reflection system in $\langle \mathcal{C} \rangle = \langle S \rangle$.
- (ii) $T := S^\times \cup ((S^\times - S^\times) \cap A^0)$ is a tame affine reflection system in $\langle \mathcal{C} \rangle = \langle S^\times \rangle = \langle T \rangle$.

Proof. (i) From the way S is defined, we have $\langle \mathcal{C} \rangle = \langle S \rangle$. Now it is immediate that (R1) and (R2) are satisfied. Next suppose $\alpha \in S^\times$ and $\beta \in S$, then for $k \in \mathbb{Z}$, $\beta + k\alpha \in S$ if and only if $\beta + k\alpha \in R$. Now as (R3) holds for R , one gets (R3) for S . Now we show that S is irreducible. Suppose that $S^\times = S_1 \uplus \dots \uplus S_t$ where S_1, \dots, S_t are connected subsets of S^\times with $(S_i, S_j) = \{0\}$ for $1 \leq i \neq j \leq t$. Since $\mathcal{C} \subseteq S^\times$ is connected, there is $1 \leq j \leq t$ such that $\mathcal{C} \subseteq S_j$. Now as for each $1 \leq k \leq t$ with $k \neq j$, $(S_k, S_j) = \{0\}$, we have $(S_k, \langle \mathcal{C} \rangle) = \{0\}$. This implies that $S^\times = \langle \mathcal{C} \rangle \cap R^\times \subseteq S_j$ which in turn gives that $t = 1$ and so S^\times is connected.

(ii) One knows that $\mathcal{W}_{S^\times} S^\times \subseteq S^\times$. Also as S is an affine reflection system in $\langle S \rangle$, one gets using Corollary 1.9 that the image of S under the canonical projection map $\langle S \rangle \rightarrow \langle S \rangle / \langle S \rangle^0$ is an irreducible locally finite root system in its \mathbb{Z} -span. Now By Proposition 1.12, $T := S^\times \cup ((S^\times - S^\times) \cap A^0) = S^\times \cup ((S^\times - S^\times) \cap \langle S \rangle^0)$ is a tame affine reflection system in $\langle S^\times \rangle = \langle T^\times \rangle = \langle T \rangle$. \square

Remark 1.16. (i) Let \dot{R} be an irreducible locally finite root system in an abelian group \dot{A} and let $R = \{S_{\dot{\alpha}}\}_{\dot{\alpha} \in \dot{R}^\times}$ be a root system extended by an abelian group G in the sense of [Y2] where one replaces “finite” with “locally finite”. Fix $\dot{\alpha} \in \dot{R}_{sh}$, $\dot{\beta} \in \dot{R}_{lg}$, if $\dot{R}_{lg} \neq \emptyset$, and $\dot{\gamma} \in \dot{R}_{ex}$, if $\dot{R}_{ex} \neq \emptyset$, and set $S := S_{\dot{\alpha}}$, $L := S_{\dot{\beta}}$ and $E := S_{\dot{\gamma}}$. Then S , L and E satisfy all conditions appearing in Theorem 1.13, and so the set R defined by $(\star\star)$ is a tame irreducible affine reflection system in $A := \dot{A} \oplus G$. Conversely, let $(A, (\cdot, \cdot), R)$ be a tame irreducible affine reflection system and let \dot{R} , S , L and E be as in the reverse part of Theorem 1.13. For $\dot{\alpha} \in \dot{R}^\times$, set $S_{\dot{\alpha}} := S$, if $\dot{\alpha} \in \dot{R}_{sh}$, $S_{\dot{\alpha}} := L$, if $\dot{\alpha} \in \dot{R}_{lg}$, and $S_{\dot{\alpha}} := E$, if $\dot{\alpha} \in \dot{R}_{ex}$. Then using Theorem 1.13, it is straightforward to see that the collection

$$R := \{S_{\dot{\alpha}}\}_{\dot{\alpha} \in \dot{R}^\times}$$

is a root system extended by the abelian group $G := A^0$.

(ii) Let $(A, (\cdot, \cdot), R)$ be a tame irreducible affine reflection system and transfer the form from A to the \mathbb{Q} -vector space $\mathcal{V} := \mathbb{Q} \otimes_{\mathbb{Z}} A$. Let \mathcal{V}^0 be the radical of the form on \mathcal{V} and set $\overline{\mathcal{V}} := \mathcal{V} / \mathcal{V}^0$. Since $A = \dot{A} \oplus A^0$, it is not difficult to see that $\mathbb{Q} \otimes A^0$ can be identified with the radical of the form on \mathcal{V} . It follows that we may naturally identify $\overline{\mathcal{V}}$ with $\mathbb{Q} \otimes \dot{A}$. Using this identification together with Corollary 1.9, we conclude that the form on $\overline{\mathcal{V}}$ is positive definite. Therefore the form on \mathcal{V} is positive semidefinite.

Suppose now that A is torsion free. Then one might identify A with the subset $1 \otimes A$ of \mathcal{V} , under the assignment $a \mapsto 1 \otimes a$, $a \in A$. Under this identification R is identified with $1 \otimes R$. It is now easy to see that the triple $(1 \otimes R, (\cdot, \cdot), \mathcal{V})$ satisfies axioms (LR1)-(LR4) of [AY, Def. 1.2], and so is an irreducible locally extended affine root system in the sense of [AY, Def. 1.2]. By [AY, Pro. 1.3], the set of non-isotropic roots of $(1 \otimes R, (\cdot, \cdot), \mathcal{V})$ is a locally extended affine root system in the sense of [Y3].

Conversely, if $(R, (\cdot, \cdot), \mathcal{V})$ is an irreducible locally extended affine root system in the sense of [AY], then it follows immediately from definition that R is a tame irreducible affine reflection system in the torsion free abelian group $A := \langle R \rangle$.

(iii) In [N1], the author defines the term “an affine reflection system” in a setting different from us. It follows from [N1, 3.11(b)] and [AY, Pro. 1.3] that a subset R of a vector space is a tame irreducible affine reflection system in the sense of [N1] if and only if it is an irreducible locally extended affine root system in the sense of [AY].

Here we introduce some terminologies and recall an elementary fact about free abelian groups which will be used frequently in the sequel. Suppose that G is an additive abelian group and p is a prime number. For a subgroup H of G with $pG \subseteq H \subseteq G$, consider G/H as a vector space over \mathbb{Z}_p . Let K be a subset of G with $H \subseteq \langle K \rangle$. We say a subset S of K is a *coset spanning set* for K in H if $\{s + H \mid s \in S\}$ spans the vector subspace $\langle K \rangle/H$. The subset S is called a *coset basis* for K in H if $\{s + H \mid s \in S\}$ is a basis for the vector subspace $\langle K \rangle/H$. We also call $\mathcal{R} \subseteq K$ a *strong coset spanning set* for K in H (with respect to G), if $K = \cup_{x \in \mathcal{R}} (x + H) \cap K$. One can easily see that any strong coset spanning set is a coset spanning set. Here is an example of a coset spanning set which is not strong: Set $G := \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$, $H := 2G$ and $K := G$. Then $\mathcal{R} := \{\sigma_1, \sigma_2\}$ is a coset spanning set for K in $2G$ which is not strong. In fact \mathcal{R} is a coset basis. The set $\mathcal{R}' = \{\sigma_1, \sigma_2, \sigma_1 + \sigma_2\}$ is a minimal strong coset spanning set for K in $2G$.

Lemma 1.17. *Suppose that p is a prime number, G a free abelian group and H a subgroup of G satisfying $pG \subseteq H \subseteq G$. Suppose that $\bar{\cdot} : G \rightarrow G/pG$ and $[\cdot] : G \rightarrow G/H$ are canonical projection maps. Let $\{\alpha_i \mid i \in I\} \subseteq H$ and $\{\beta_j \mid j \in J\} \subseteq G$ be such that $\{\bar{\alpha}_i \mid i \in I\}$ is a linearly independent subset of the \mathbb{Z}_p -vector subspace H/pG of G/pG and $\{[\beta_j] \mid j \in J\}$ is a linearly independent subset of the quotient space G/H . Then $\{\alpha_i, \beta_j \mid i \in I, j \in J\}$ is a \mathbb{Z} -linearly independent subset of the free abelian group G .*

Let $(A, (\cdot, \cdot), R)$ be a tame irreducible affine reflection system with Weyl group \mathcal{W} . As we have seen in Theorem 1.13, we have $A = \dot{A} \oplus A^0$, where \dot{A} is a free abelian subgroup of A and A^0 is the radical of the form. Let

$$\mathfrak{p} : A \rightarrow A^0 \quad (1.18)$$

be the canonical projection map from A onto A^0 .

For a subset \mathcal{P} of R^\times , considering (1.2), we set

$$\mathcal{P}_{sh} := \mathcal{P} \cap R_{sh} \quad \text{and} \quad \mathcal{P}_{lg} := \mathcal{P} \cap R_{lg}.$$

Definition 1.19. Let $\mathcal{P} \subseteq R^\times$.

- (i) We call \mathcal{P} , a *reflectable set* for R , if $\mathcal{W}_{\mathcal{P}}\mathcal{P} = R^\times$.
- (ii) The subset \mathcal{P} is called a *reflectable base* if it is a reflectable set and no proper subset of \mathcal{P} is a reflectable set.
- (iii) If A is a free abelian group, the subset \mathcal{P} is called an *integral base* for R if \mathcal{P} is a basis for A .

(iv) If A is a free abelian group, we call \mathcal{P} an *integral reflectable base* if \mathcal{P} is a reflectable base which is an integral base as well.

We remark that if Π is a nonempty subset of R^\times and $\alpha \in \Pi$, then $\pm\alpha \in \mathcal{W}_\Pi\Pi$ and that Π is a reflectable set (resp. reflectable base) if and only if $(\Pi \setminus \{\alpha\}) \cup \{-\alpha\}$ is a reflectable set (resp. reflectable base). We refer to the latter property as the *sign freeness* condition of reflectable sets (resp. reflectable bases). We also remark that by Proposition 1.7 and [LN2, Lemma 5.1], any reduced locally finite root system possesses an integral reflectable base. In [MS] and [A2], a reflectable base is constructed for any extended affine root system of reduced type.

Lemma 1.20. *Any reflectable set for R is a connected generating set for $\langle R \rangle$.*

Proof. Let \mathcal{P} be a reflectable set. Clearly we have $\langle R \rangle = \langle R^\times \rangle = \langle \mathcal{W}_\mathcal{P}\mathcal{P} \rangle \subseteq \langle \mathcal{P} \rangle \subseteq \langle R \rangle$ and so \mathcal{P} generates $\langle R \rangle$.

Next suppose that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1, \mathcal{P}_2$ are two nonempty subsets with $(\mathcal{P}_1, \mathcal{P}_2) = \{0\}$. Then $\mathcal{W}_\mathcal{P}\mathcal{P}_i \subseteq \langle \mathcal{P}_i \rangle$ for $i = 1, 2$. So $R^\times = \mathcal{W}_\mathcal{P}\mathcal{P} = \mathcal{W}_\mathcal{P}\mathcal{P}_1 \cup \mathcal{W}_\mathcal{P}\mathcal{P}_2$ and $(\mathcal{W}_\mathcal{P}\mathcal{P}_1, \mathcal{W}_\mathcal{P}\mathcal{P}_2) \subseteq (\langle \mathcal{P}_1 \rangle, \langle \mathcal{P}_2 \rangle) = \{0\}$. This implies that R^\times is disconnected, a contradiction. \square

Lemma 1.21. *Let $\Pi \subseteq R^\times$.*

(i) *If $\mathcal{P} \subseteq \Pi$, $\beta \in \Pi \setminus \mathcal{P}$ and Π' is the set obtained from Π by replacing β with any element of the orbit $\mathcal{W}_\mathcal{P} \cdot \beta$. Then $\mathcal{W}_\Pi\Pi = \mathcal{W}_{\Pi'}\Pi'$. In particular Π is a reflectable set if and only if Π' is a reflectable set.*

(ii) *If $\alpha, \alpha - \mathbf{p}(\alpha) \in \Pi$, then Π is a reflectable set if and only if $\Pi' := (\Pi \setminus \{\alpha\}) \cup \{\alpha - 2\mathbf{p}(\alpha)\}$ is a reflectable set.*

Proof. (i) Take $\alpha := w'\beta$ for some $w' \in \mathcal{W}_\mathcal{P}$ and $\Pi' := (\Pi \setminus \{\beta\}) \cup \{\alpha\}$. We show that $\mathcal{W}_{\Pi'} = \mathcal{W}_\Pi$. For this, it is enough to show that $w_\gamma \in \mathcal{W}_{\Pi'}$ for any $\gamma \in \Pi$. If $\gamma \neq \beta$, then $\gamma \in \Pi'$ and so $w_\gamma \in \mathcal{W}_{\Pi'}$. If $\gamma = \beta$, then $w_\gamma = (w')^{-1}w_\alpha w' \in \mathcal{W}_\mathcal{P}\mathcal{W}_{\Pi'}\mathcal{W}_\mathcal{P} \subseteq \mathcal{W}_{\Pi'}$. Similarly, we have $\mathcal{W}_{\Pi'} \subseteq \mathcal{W}_\Pi$. This completes the proof.

(ii) Set $\mathcal{P} = \Pi \setminus \{\alpha\}$. Then $-\alpha + 2\mathbf{p}(\alpha) = w_{\alpha - \mathbf{p}(\alpha)}(\alpha) \in \mathcal{W}_\mathcal{P} \cdot \alpha$. So by Part (i), with $\beta = \alpha$, we get Π is a reflectable set if and only if $(\Pi \setminus \{\alpha\}) \cup \{-\alpha + 2\mathbf{p}(\alpha)\}$ is a reflectable set. But then by sign freeness, the latter is a reflectable set if and only if Π' is a reflectable set, as required. \square

Lemma 1.22. *Let R be of one of non-simply laced types and $\alpha, \alpha_1, \dots, \alpha_n \in R^\times$. Suppose that $\{\alpha_{i_1}, \dots, \alpha_{i_t}\} = \{\alpha_1, \dots, \alpha_n\} \cap R_{sh}$ and $\{\alpha_{j_1}, \dots, \alpha_{j_m}\} = \{\alpha_1, \dots, \alpha_n\} \cap R_{lg}$. Then*

$$w_{\alpha_1} \cdots w_{\alpha_n}(\alpha) \in \begin{cases} w_{\alpha_{i_1}} \cdots w_{\alpha_{i_t}}(\alpha) + \langle R_{lg} \rangle & \text{if } \alpha \in R_{sh} \\ w_{\alpha_{j_1}} \cdots w_{\alpha_{j_m}}(\alpha) + \rho \langle R_{sh} \rangle & \text{if } \alpha \in R_{lg}. \end{cases}$$

Proof. Let $\alpha, \beta \in R_{lg}$ and $\gamma \in R_{sh}$. We know that $(\alpha, \gamma^\vee) \in \rho\mathbb{Z}$. Therefore $w_\gamma(\alpha) \in \alpha + \rho \langle R_{sh} \rangle$. Then

$$w_\gamma w_\beta(\alpha) = w_\beta(\alpha) + \rho \langle R_{sh} \rangle.$$

Now it follows using an inductive process that $w_{\alpha_1} \cdots w_{\alpha_n}(\alpha) \in w_{\alpha_{j_1}} \cdots w_{\alpha_{j_m}}(\alpha) + \rho\langle R_{sh} \rangle$. If $\alpha \in R_{sh}$, then an analogous argument as above, using the fact that $(\beta, \alpha^\vee) \in \mathbb{Z}$ if $\beta \in R_{lg}$, gives the other implication. \square

Note that if R is of one of non-simply laced types, it follows from Theorem 1.13 and the known facts on locally finite root systems (see Proposition 1.7 and [LN1]) that

$$\rho\langle R_{sh} \rangle \subseteq \langle R_{lg} \rangle \subseteq \langle R_{sh} \rangle.$$

Therefore,

$$\langle R_{sh} \rangle / \langle R_{lg} \rangle \text{ and } \langle R_{lg} \rangle / \rho\langle R_{sh} \rangle \text{ are two vector spaces over } \mathbb{Z}_\rho. \quad (1.23)$$

Lemma 1.24. *Let R be of one of non-simply laced types and Π be a reflectable set for R . Then Π contains a subset \mathcal{M} such that $\mathcal{M}_{sh} = \mathcal{M} \cap R_{sh}$ (resp. $\mathcal{M}_{lg} = \mathcal{M} \cap R_{lg}$) is a coset basis for R_{sh} in $\langle R_{lg} \rangle$ (resp. for R_{lg} in $\rho\langle R_{sh} \rangle$). Moreover, we have*

- (i) $\{\mathfrak{p}(\alpha) \mid \alpha \in \mathcal{M}^{lg}\}$ is a coset spanning set for L in $\rho\langle S \rangle$,
- (ii) $\{\mathfrak{p}(\alpha) \mid \alpha \in \mathcal{M}^{sh}\}$ is a coset spanning set for S in $\langle L \rangle$.

Proof. From Lemma 1.22, it follows that, as subsets of the abelian group $\langle R_{lg} \rangle / \rho\langle R_{sh} \rangle$, we have

$$\begin{aligned} \langle R_{lg} \rangle / \rho\langle R_{sh} \rangle &= \langle \mathcal{W}_\Pi \Pi_{lg} \rangle / \rho\langle R_{sh} \rangle \\ &= \langle \{w\alpha + \rho\langle R_{sh} \rangle \mid w \in \mathcal{W}_\Pi, \alpha \in \Pi_{lg}\} \rangle \\ &= \langle \{\alpha + \rho\langle R_{sh} \rangle \mid \alpha \in \Pi_{lg}\} \rangle. \end{aligned}$$

Therefore Π_{lg} is a coset spanning set for R_{lg} in $\rho\langle R_{sh} \rangle$. So Π_{lg} contains a coset basis P_1 for R_{lg} in $\rho\langle R_{sh} \rangle$. An analogous argument shows that Π_{sh} contains a coset basis P_2 for R_{sh} in $\langle R_{lg} \rangle$. Now the set $\mathcal{M} := P_1 \cup P_2$ satisfies the first assertion of the statement with $\mathcal{M}_{sh} = P_1$ and $\mathcal{M}_{lg} = P_2$.

Next, let $\delta \in L$. Then

$$\delta + \rho\langle R_{sh} \rangle \in \sum_{\alpha \in \mathcal{M}_{lg}} \mathbb{Z}\alpha + \rho\langle R_{sh} \rangle = \sum_{\alpha \in \mathcal{M}_{lg}} \mathbb{Z}(\alpha - \mathfrak{p}(\alpha) + \mathfrak{p}(\alpha)) + \rho\langle R_{sh} \rangle + \rho\langle S \rangle.$$

From this it follows that $\delta \in \sum_{\alpha \in \mathcal{M}_{lg}} (\mathfrak{p}(\alpha) + \rho\langle S \rangle)$, showing that Part (i) holds. The argument for Part (ii) is similar. \square

2. Characterization of reflectable bases for locally finite root systems

In this section, we characterize reflectable bases for locally finite root systems. This will be essential in obtaining the characterization theorem for the general case. Let \dot{R} be a locally finite root system of the form (1.10). For $\dot{\alpha} = \sum_{i \in I} \eta_i \epsilon_i \in \dot{R}$, we set

$$\text{supp}(\dot{\alpha}) := \{i \in I \mid \eta_i \neq 0\}.$$

Also for a subset \dot{N} of \dot{R}^\times , we set

$$\text{supp}(\dot{N}) := \cup_{\dot{\alpha} \in \dot{N}} \text{supp}(\dot{\alpha}).$$

Lemma 2.1. *Let \dot{R} be an irreducible locally finite root system of type $X = \dot{A}$, D or B_I ($|I| \geq 3$) and \mathcal{P} be a coset spanning set for \dot{R}^\times in $2\langle \dot{R} \rangle$ if $X = \dot{A}, D$ and a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ if $X = B$. Then $\text{supp}(\mathcal{P}) = I$ and \mathcal{P} is connected.*

Proof. The first assertion in the statement is easily checked. So we just need to show the second assertion. We carry out this in the following two steps:

Step 1. If \mathcal{P} is disconnected, then there are two nonempty subsets $\mathcal{P}_1, \mathcal{P}_2$ of \mathcal{P} with $\text{supp}(\mathcal{P}_1) \cap \text{supp}(\mathcal{P}_2) = \emptyset$ such that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Indeed, we claim that there exists either $\alpha \in \mathcal{P}_1$ such that $\text{supp}(\alpha) \neq \text{supp}(\beta)$ for all $\beta \in \mathcal{P}_2$, or $\beta \in \mathcal{P}_2$ such that $\text{supp}(\beta) \neq \text{supp}(\alpha)$ for all $\alpha \in \mathcal{P}_1$. Suppose this does not hold, then for any $\alpha \in \mathcal{P}_1$, there exists a unique $\bar{\alpha} \in \mathcal{P}_2$ such that $\text{supp}(\alpha) = \text{supp}(\bar{\alpha})$ and $(\alpha, \bar{\alpha}) = 0$. Similarly, for any $\beta \in \mathcal{P}_2$, there exists a unique element $\bar{\beta} \in \mathcal{P}_1$ such that $\text{supp}(\beta) = \text{supp}(\bar{\beta})$ and $(\beta, \bar{\beta}) = 0$. Now if $\alpha_1, \alpha_2 \in \mathcal{P}_1$ and $\text{supp}(\alpha_1) \cap \text{supp}(\alpha_2) = \{i\}$ for some $i \in I$, then $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{P}_2$ and $\{i\} = \text{supp}(\bar{\alpha}_1) \cap \text{supp}(\bar{\alpha}_2)$, so $(\alpha_1, \bar{\alpha}_2) \neq 0$ which contradicts the fact that $(\mathcal{P}_1, \mathcal{P}_2) = \{0\}$. This contradiction shows that for each $i \in I$, there exists a unique $\alpha \in \mathcal{P}_1$ such that $i \in \text{supp}(\alpha)$, and in this case $\bar{\alpha}$ is the unique element in \mathcal{P}_2 such that $i \in \bar{\alpha}$. Since the \mathbb{Z}_2 -vector spaces $\langle \dot{R} \rangle / 2\langle \dot{R} \rangle$ (if $X = A, D$) and $\langle \dot{R}_{lg} \rangle / 2\langle \dot{R}_{sh} \rangle$ are of dimension greater than or equal 2 and \mathcal{P} is a coset spanning set, one finds $\beta \in \mathcal{P}_1 \setminus \{\alpha\}$. Let $\text{supp}\{\alpha\} = \{i_\alpha, j_\alpha\}$ and $\text{supp}(\beta) = \{i_\beta, j_\beta\}$. Then as we have seen above, α is the unique element in \mathcal{P}_1 having i_α or j_α in its support. Similarly β is the unique element in \mathcal{P}_1 having i_β or j_β in its support. Also since B is a coset spanning set for \dot{R} in $2\dot{A}$, we have $\epsilon_{i_\alpha} + \epsilon_{i_\beta} \in \langle \mathcal{P}_1 \rangle + \langle \mathcal{P}_2 \rangle + 2\dot{A}$. Now define $f : \sum_{i \in I} \mathbb{Z}\epsilon_i \rightarrow \mathbb{Z}_2$ by $f(\epsilon_{i_\alpha}) = f(\epsilon_{j_\alpha}) = 1$ and $f(\epsilon_i) = 0$ for all $i \in I \setminus \{i_\alpha, j_\alpha\}$. Then $f(\epsilon_{i_\alpha} + \epsilon_{i_\beta}) = 1$ but $f(\langle \mathcal{P}_1 \rangle) = \{0\}$, $f(\langle \mathcal{P}_2 \rangle) = \{0\}$ and $f(2\dot{A}) = \{0\}$, a contradiction. This proves that our claim is true, so without loss of generality, we may assume that

$$\text{there exists } \beta \in \mathcal{P}_2 \text{ such that } \text{supp}(\beta) \neq \text{supp}(\alpha) \text{ for all } \alpha \in \mathcal{P}_1. \quad (2.2)$$

Now set

$$\begin{aligned} \mathcal{P}' &:= \{\beta \in \mathcal{P}_2 \mid \text{supp}(\beta) = \text{supp}(\alpha) \text{ for some } \alpha \in \mathcal{P}_1\}, \\ \mathcal{P}'_1 &:= \mathcal{P}_1 \cup \mathcal{P}' \quad \text{and} \quad \mathcal{P}'_2 := \mathcal{P}_2 \setminus \mathcal{P}'. \end{aligned}$$

Then using (2.2), we have

$$\mathcal{P} = \mathcal{P}'_1 \cup \mathcal{P}'_2, \quad \mathcal{P}'_1 \neq \emptyset, \quad \mathcal{P}'_2 \neq \emptyset, \quad (\mathcal{P}'_1, \mathcal{P}'_2) = \{0\} \quad \text{and} \quad \text{supp}(\mathcal{P}'_1) \cup \text{supp}(\mathcal{P}'_2) = \emptyset.$$

So replacing \mathcal{P}_i with \mathcal{P}'_i , $i = 1, 2$, if necessary, we may assume that $\text{supp}(\mathcal{P}_1) \cap \text{supp}(\mathcal{P}_2) = \emptyset$.

Step 2. \mathcal{P} is connected: Suppose to the contrary that \mathcal{P} is not connected, then by Step 1, there are nonempty subsets $\mathcal{P}_1, \mathcal{P}_2$ of \mathcal{P} with $\text{supp}(\mathcal{P}_1) \cap \text{supp}(\mathcal{P}_2) = \emptyset$ such that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Fix $u \in \text{supp}(\mathcal{P}_1)$ and $v \in \text{supp}(\mathcal{P}_2)$. Take $G := \langle \dot{R} \rangle$ if $X = A, D$ and $G := \langle \dot{R}_{sh} \rangle$ if $X = B$. Now define $f : \sum_{i \in I} \mathbb{Z}\epsilon_i \rightarrow \mathbb{Z}_2$ such that $f(\epsilon_i) = 1$ for all $i \in \text{supp}(\mathcal{P}_1)$ and $f(\epsilon_i) = 0$ for all $i \in \text{supp}(\mathcal{P}_2)$. Since \mathcal{P} is a coset spanning set, we have $\epsilon_u - \epsilon_v \in \langle \mathcal{P}_1 \rangle + \langle \mathcal{P}_2 \rangle + 2G$ and so $f(\epsilon_u - \epsilon_v) = 1$, $f(\langle \mathcal{P}_1 \rangle) = \{0\}$, $f(\langle \mathcal{P}_2 \rangle) = \{0\}$ and $f(2G) = \{0\}$, a contradiction. This completes the proof. \square

Type B_I : Suppose \dot{R} is a locally finite root system of type B_I .

Proposition 2.3. *Suppose that $|I| = 2$. A subset $\dot{\Pi}$ of \dot{R}^\times is a reflectable set if and only if $\dot{\Pi} \cap \dot{R}_{sh} \neq \emptyset$ and $\dot{\Pi} \cap \dot{R}_{lg} \neq \emptyset$. Moreover a reflectable set $\dot{\Pi}$ is a reflectable base if and only if $|\dot{\Pi}| = 2$.*

Proof. It is easy to check. \square

Proposition 2.4. *Let $|I| \geq 3$.*

(i) *If $\dot{\mathcal{M}} \subseteq \dot{R}_{lg}$ is a coset spanning set for \dot{R}_{lg} in $\langle 2\dot{R}_{sh} \rangle$ and $i, j \in I$ with $i \neq j$, then there exist $w \in \mathcal{W}_{\dot{\mathcal{M}}}$ and $\dot{\alpha} \in \dot{\mathcal{M}}$ such that $\{i, j\} = \text{supp}(w\dot{\alpha})$.*

(ii) *Suppose that $\dot{\Pi}$ is a reflectable set for \dot{R} . Then $\dot{\Pi}_{lg}$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$.*

(iii) *Let $\dot{\mathcal{M}} \subseteq \dot{R}_{lg}$ be a coset spanning set for \dot{R}_{lg} in $\langle 2\dot{R}_{sh} \rangle$ and $\dot{\alpha} \in \dot{R}_{sh}$. Then $\dot{\Pi} := \{\dot{\alpha}\} \cup \dot{\mathcal{M}}$ is a reflectable set for \dot{R} . Moreover if $\dot{\mathcal{M}}$ is a coset basis, then $\dot{\Pi}$ is a reflectable base for \dot{R} .*

(iv) *Suppose that $\dot{\Pi}$ is a reflectable base for \dot{R} . Then $\dot{\Pi}$ contains a short root $\dot{\alpha}$ and a coset basis $\dot{\mathcal{M}}$ of \dot{R}_{lg} in $\langle 2\dot{R}_{sh} \rangle$ such that $\dot{\Pi} = \{\dot{\alpha}\} \cup \dot{\mathcal{M}}$.*

Proof. (i) By Lemma 2.1, $\dot{\mathcal{M}}$ is connected and $\text{supp}(\dot{\mathcal{M}}) = I$. If $\dot{\mathcal{M}}$ contains an element of support $\{i, j\}$, there is nothing to prove. Also, if there exist $t \in I$ and $\dot{\beta}, \dot{\gamma} \in \dot{\mathcal{M}}$ with $\text{supp}(\dot{\beta}) = \{i, t\}$ and $\text{supp}(\dot{\gamma}) = \{j, t\}$, then $\{i, j\} = \text{supp}(w_{\dot{\beta}}(\dot{\gamma}))$ and we are done. Otherwise, since $\dot{\mathcal{M}}$ is connected and $\text{supp}(\dot{\mathcal{M}}) = I$, we may find $\dot{\alpha}_0, \dots, \dot{\alpha}_n \in \dot{\mathcal{M}}$ such that $\text{supp}(\dot{\alpha}_0) = \{i, s_0\}$, $\text{supp}(\dot{\alpha}_n) = \{s_{n-1}, j\}$ and $\text{supp}(\dot{\alpha}_k) = \{s_{k-1}, s_k\}$ for $k = 1, \dots, n-1$. Then $\text{supp}(w_{\dot{\alpha}_0} \cdots w_{\dot{\alpha}_{n-1}}(\dot{\alpha}_n)) = \{i, j\}$, as required.

(ii) See Lemma 1.24.

(iii) Assume that $\dot{\alpha} = \eta_0 \epsilon_{i_0}$ for some $i_0 \in I$, $\eta_0 \in \{\pm 1\}$. By Part (i), for any $j \in I \setminus \{i_0\}$, there are $\eta'_0, \eta_j \in \{\pm 1\}$ such that $\eta'_0 \epsilon_{i_0} + \eta_j \epsilon_j \in \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$. Set $\mathcal{P}' := \{\epsilon_{i_0}, \epsilon_{i_0} - \epsilon_j \mid j \in I\} \subseteq \mathcal{W}_{\dot{\Pi}} \dot{\Pi}$. Then from [LN2, Lemma 5.1], it follows that

$$\dot{R}^\times = \mathcal{W}_{\mathcal{P}'} \mathcal{P}' \subseteq \mathcal{W}_{\dot{\Pi}} \dot{\Pi} \subseteq \dot{R}^\times$$

which shows that $\mathcal{W}_{\dot{\Pi}} \dot{\Pi} = \dot{R}^\times$, i.e., $\dot{\Pi}$ is a reflectable set. Now suppose $\dot{\mathcal{M}}$ is a coset basis. We show that $\dot{\Pi}$ is a reflectable base. Suppose that $\mathcal{P} \subseteq \dot{\Pi}$ is a reflectable set, then by Part (ii), $\mathcal{P}_{lg} \subseteq \dot{\mathcal{M}}$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and so by minimality of $\dot{\mathcal{M}}$ (see Part (i)), we get that $\mathcal{P}_{lg} = \dot{\Pi}_{lg} = \dot{\mathcal{M}}$ which in turn implies that $\mathcal{P} = \dot{\Pi}$.

(iv) Since $\dot{\Pi}$ is a reflectable base, $\dot{\Pi}_{lg}$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ by Part (ii). Fix $\dot{\alpha} \in \dot{\Pi}_{sh}$, then by Part (iii), $\mathcal{P} := \{\dot{\alpha}\} \cup \dot{\Pi}_{lg}$ is a reflectable set. Now the minimality of $\dot{\Pi}$ implies that $\dot{\Pi}_{sh} = \{\dot{\alpha}\}$. Next, we claim that $\dot{\Pi}_{lg}$ is a coset basis for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$, indeed if it is not the case, then there is a coset spanning set $\dot{\mathcal{P}} \subsetneq \dot{\Pi}_{lg}$ for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$. Now using Part (iii), $\{\dot{\alpha}\} \cup \dot{\mathcal{P}}$ is a reflectable set for \dot{R} . This contradicts the minimality of $\dot{\Pi}$ and so we are done. \square

Types $\dot{A}_I(|I| \geq 2), \dot{D}_I(|I| \geq 4)$: Suppose that \dot{R} is a locally finite root system of the types under consideration. If \dot{R} is of type A_1 , then $\dot{R}^\times = \{\pm \dot{\alpha}\}$. In this case, $\{\dot{\alpha}\}$, $\{-\dot{\alpha}\}$ and $\{\pm \dot{\alpha}\}$ are the only reflectable sets and $\{\dot{\alpha}\}$, $\{-\dot{\alpha}\}$ are the only reflectable bases. So from now on we assume \dot{R} is a locally finite root system of type $\dot{A}_I(|I| \geq 3)$ or $\dot{D}_I(|I| \geq 4)$.

Proposition 2.5. (a) Let $\dot{\mathcal{M}} \subseteq \dot{R}^\times$ be a generating set for $\langle \dot{R} \rangle$, then we have the followings:

- (i) $\dot{\mathcal{M}}$ is connected.
 - (ii) For $i_0, j_0 \in I$ with $i_0 \neq j_0$, there exist $\dot{\alpha} \in \dot{\mathcal{M}}$ and $w \in \mathcal{W}_{\dot{\mathcal{M}}}$ such that $\text{supp}(w\dot{\alpha}) = \{i_0, j_0\}$.
 - (iii) If \dot{R} is of type D_I , then there exist $\dot{\alpha}, \dot{\beta} \in \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$ such that $\text{supp}(\dot{\alpha}) = \text{supp}(\dot{\beta})$ and $(\dot{\alpha}, \dot{\beta}) = 0$.
 - (iv) $\dot{\mathcal{M}}$ is a reflectable set.
- (b) $\dot{\Pi} \subseteq \dot{R}^\times$ is a reflectable base for \dot{R} if and only if $\dot{\Pi}$ is a minimal generating set for $\langle \dot{R} \rangle$.

Proof. (a)(i) Since any subset of \dot{R}^\times generating $\langle \dot{R} \rangle$ is a spanning set for \dot{R}^\times in $2\langle \dot{R} \rangle$, we are done using Lemma 2.1.

(a)(ii) Fix $i_0, j_0 \in I$ with $i_0 \neq j_0$. If $\dot{\mathcal{M}}$ contains an element whose support is $\{i_0, j_0\}$, there is nothing to prove, otherwise set $\dot{\mathcal{M}}(i) := \{j \in I \mid \{i, j\} = \text{supp}(\dot{\alpha}) \text{ for some } \dot{\alpha} \in \dot{\mathcal{M}}\}$, $i \in I$. Since $\dot{\mathcal{M}}$ generates the abelian group $\langle \dot{R} \rangle$, it follows that $\dot{\mathcal{M}}(i) \neq \emptyset$ for all $i \in I$. Now this together with the connectedness of $\dot{\mathcal{M}}$ (Part (a)(i)) implies that there are $\dot{\alpha}, \dot{\beta} \in \dot{\mathcal{M}}$ and $\dot{\alpha}_0 := \dot{\alpha}, \dots, \dot{\alpha}_n := \dot{\beta}$ in $\dot{\mathcal{M}}$ such that $\text{supp}(\dot{\alpha}) \cap \dot{\mathcal{M}}(i_0) \neq \emptyset$, $\text{supp}(\dot{\beta}) \cap \dot{\mathcal{M}}(j_0) \neq \emptyset$, $\text{supp}(\dot{\alpha}_0) = \{i_0, i\}$, $\text{supp}(\dot{\alpha}_n) = \{j, j_0\}$ and

$$\text{supp}(\dot{\alpha}_1) = \{i, j_1\}, \text{supp}(\dot{\alpha}_2) = \{j_1, j_2\}, \dots, \text{supp}(\dot{\alpha}_{n-1}) = \{j_{n-1}, j\}.$$

Then $\{i_0, j_0\} = \text{supp}(w_{\dot{\alpha}_n} w_{\dot{\alpha}_{n-1}} \cdots w_{\dot{\alpha}_1}(\dot{\alpha}_0))$. This completes the proof.

(a)(iii) Suppose to the contrary that there not exist $\dot{\alpha}, \dot{\beta} \in \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$ with $\text{supp}(\dot{\alpha}) = \text{supp}(\dot{\beta})$ and $(\dot{\alpha}, \dot{\beta}) = 0$. Then fix $i_0 \in I$. It follows from Part (a)(ii) that for any $i \in I \setminus \{i_0\}$, there is $r_i \in \{\pm 1\}$ such that $\epsilon_{i_0} + r_i \epsilon_i \in \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$. This implies that for $i, j \in I \setminus \{i_0\}$ with $i \neq j$, $r_i \epsilon_i - r_j \epsilon_j \in \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$ and so one gets that

$$\mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}} = \{\pm(\epsilon_{i_0} + r_i \epsilon_i) \mid i \in I \setminus \{i_0\}\} \cup \{\pm(r_i \epsilon_i - r_j \epsilon_j) \mid i, j \in I \setminus \{i_0\}, i \neq j\}.$$

Now define $f : \sum_{i \in I} \mathbb{Z} \epsilon_i \longrightarrow \mathbb{Z}_3$ by

$$\epsilon_{i_0} \mapsto 1, \quad \epsilon_i \mapsto 2r_i; \quad i \in I \setminus \{i_0\}.$$

Then $f(\langle \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}} \rangle) = \{0\}$ which is a contradiction as $\epsilon_{i_0} - r_i \epsilon_i \in \dot{R}_{lg} \subseteq \langle \dot{\mathcal{M}} \rangle \subseteq \langle \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}} \rangle$ and $f(\epsilon_{i_0} - r_i \epsilon_i) = 2$. This completes the proof.

(a)(iv) Let $i, j \in I$ with $i \neq j$. By Part (a)(ii), there exist $\dot{\alpha} \in \dot{\mathcal{M}}$ and $w \in \mathcal{W}_{\dot{\mathcal{M}}}$ such that $\text{supp}(w\dot{\alpha}) = \{i, j\}$. Now if we are in case \dot{A}_I , then the only roots with support $\{i, j\}$ are $\pm(\epsilon_i - \epsilon_j)$, and so we are done in this case.

Next suppose we are in case D_I . By Part (a)(iii), $\mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$ contains at least two roots of the forms $\dot{\beta}_1 = \epsilon_i - \epsilon_j$ and $\dot{\beta}_2 = \epsilon_i + \epsilon_j$. Now for any $s \in I \setminus \{i, j\}$, by Part (a)(ii), $\mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$ contains either $\epsilon_s - \epsilon_i$ or $\epsilon_s + \epsilon_i$. But by acting $w_{\dot{\beta}_1}$ and $w_{\dot{\beta}_2}$ on the one which belongs to $\mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$, we get $\pm(\epsilon_s \pm \epsilon_j) \in \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$. Since $s \in I \setminus \{i, j\}$ was chosen arbitrary, it follows that $\pm(\epsilon_s \pm \epsilon_t) \in \mathcal{W}_{\dot{\mathcal{M}}} \dot{\mathcal{M}}$ for all $s, t \in I$ with $s \neq t$.

(b) It follows immediately from Lemma 1.20 and Part (a)(iv). \square

Proposition 2.6. *Let \dot{R} be an irreducible locally finite root system of type \dot{A}_I , $|I| \geq 2$, and $\dot{\mathcal{P}}$ be a subset of \dot{R}^\times . Then $\dot{\mathcal{P}}$ is a reflectable set (reflectable base) for \dot{R} if and only if $\dot{\mathcal{P}}$ is a coset spanning set (coset basis) for \dot{R}^\times in $2\langle\dot{R}\rangle$. In particular, any reflectable set contains a reflectable base. Moreover, any reflectable base is an integral base.*

Proof. Let $\dot{\mathcal{P}}$ be a coset spanning set for \dot{R}^\times in $2\langle\dot{R}\rangle$. Then $\dot{\mathcal{P}}$ contains a coset basis B of \dot{R}^\times in $2\langle\dot{R}\rangle$. Clearly $\text{supp}(B) = I$ and so by Lemma 2.1, B is connected. Then $S := (W_B B) \cup \{0\}$ is an irreducible locally finite root system in $\langle B \rangle = \langle S \rangle$ of the same type as \dot{R} . Now let $i_0, j_0 \in I$ with $i_0 \neq j_0$. Since $\text{supp}(S) = \text{supp}(B) = I$, there exist $\dot{\alpha}, \dot{\beta} \in S$ with $i_0 \in \text{supp}(\dot{\alpha})$ and $j_0 \in \text{supp}(\dot{\beta})$. Also since S is connected, we may find elements $\dot{\alpha}_0 := \dot{\alpha}, \dot{\alpha}_1, \dots, \dot{\alpha}_{n-1}, \dot{\alpha}_n := \dot{\beta}$ in S such that $(\dot{\alpha}_i, \dot{\alpha}_j) \neq 0$ if and only if $|i - j| \leq 1$. Then $\pm w_{\dot{\alpha}_n} \cdots w_{\dot{\alpha}_1} \dot{\alpha}_0 = \pm(\epsilon_{i_0} - \epsilon_{j_0})$. This shows that $\pm(\epsilon_{i_0} - \epsilon_{j_0}) \in S$. Now since i_0 and j_0 were chosen arbitrary, we get $S = \dot{R}$. Thus B and so $\dot{\mathcal{P}}$ is a reflectable set. Moreover, B is a reflectable base since otherwise it contains a proper subset B' which is a reflectable set. But then $\langle B' \rangle = \langle B \rangle = \langle \dot{R} \rangle$, and so B' , which is a coset spanning set for \dot{R}^\times in $2\langle\dot{R}\rangle$, is properly contained in the coset basis B , a contradiction.

Conversely, suppose $\dot{\mathcal{P}}$ is a reflectable set (reflectable base). Then $\langle \dot{\mathcal{P}} \rangle = \langle \dot{R} \rangle$ and so $\dot{\mathcal{P}}$ is a coset spanning set for \dot{R}^\times in $2\langle\dot{R}\rangle$. If $\dot{\mathcal{P}}$ is a reflectable base and $\dot{\mathcal{P}}$ is not a coset basis, then $\dot{\mathcal{P}}$ contains a proper subset B which is a coset basis for \dot{R}^\times in $2\langle\dot{R}\rangle$. But by what we have seen above, B is a reflectable base. This contradicts the minimality of $\dot{\mathcal{P}}$. The last assertion in the statement follows from the first assertion and Lemma 1.17. \square

Lemma 2.7. *Let \dot{R} be an irreducible locally finite root system of type D , and S be a subsystem of \dot{R} of type D such that $\text{supp}(S) = \text{supp}(\dot{R})$. Then $S = \dot{R}$.*

Proof. By assumption, we may assume there exists an index set K such that

$$S^\times = \{\pm(\epsilon'_t \pm \epsilon'_k) \mid t \neq k \in K\} \subseteq \dot{R}^\times = \{\pm(\epsilon_i \pm \epsilon_j) \mid i \neq j \in I\}.$$

Fix two distinct $u, v \in K$ and set

$$B' := \{\epsilon'_u - \epsilon'_t \mid t \in K \setminus \{u\}\} \cup \{\epsilon'_u + \epsilon'_v\}.$$

Then one knows that B' is a basis for the free abelian group $\langle S \rangle$. Now let $t, t' \in K \setminus \{u\}$ and $t \neq t'$. Since $S \subseteq \dot{R}$, we have $\dot{\alpha} := \epsilon'_u - \epsilon'_t = r\epsilon_m + s\epsilon_n$ and $\dot{\beta} := \epsilon'_u - \epsilon'_{t'} = r'\epsilon_l + s'\epsilon_p$ for some $r, s, r', s' \in \{\pm 1\}$ and $m, n, l, p \in I$. Since $(\dot{\alpha}, \dot{\beta}) \neq 0$ and $\dot{\alpha}, \dot{\beta}$ are \mathbb{Z} -linearly independent, one sees that the set $\{m, n, l, p\}$ is of cardinality 3. So, we may assume that $\dot{\alpha} = r\epsilon_m + s\epsilon_n$ and $\dot{\beta} = r'\epsilon_m + s'\epsilon_p$ with $p \neq n$. Now since any other element of B' of the form $\epsilon'_u - \epsilon'_t$ must be non-orthogonal to both $\dot{\alpha}$ and $\dot{\beta}$, we conclude that, we may assume $K \subseteq I$, $u = m$ and

$$B'' := \{\epsilon'_u - \epsilon'_t \mid t \in K \setminus \{u\}\} = \{r_t\epsilon_u + s_t\epsilon_t \mid t \in K \setminus \{u\}\},$$

where $r_t, s_t \in \{\pm 1\}$. Next, consider $\dot{\gamma} := \epsilon'_u + \epsilon'_v \in S$. We know that $\dot{\gamma}$ is orthogonal to none of elements of B'' except $\epsilon'_u - \epsilon'_v = r_v\epsilon_u + s_v\epsilon_v$. Therefore $\dot{\gamma} \in \pm(r_v\epsilon_u - s_v\epsilon_v)$. So without loss of generality, we may assume that

$$B' = \{r_t\epsilon_u + s_t\epsilon_t \mid t \in K \setminus \{u\}\} \cup \{r_v\epsilon_u - s_v\epsilon_v\}.$$

Therefore $K = \text{supp}(B') = \text{supp}(S) = \text{supp}(\dot{R}) = I$. Moreover for any $t \in I = K$, we have

$$\pm(\epsilon_v \pm \epsilon_t) = w_{r_t \epsilon_u + s_t \epsilon_t}(\pm(\epsilon_u \pm \epsilon_v)) \in S.$$

Since this holds for any $t \in I$, it follows again from the Weyl group action that $\pm(\epsilon_i \pm \epsilon_j) \in S$ for all $i, j \in I$. Thus $S = \dot{R}$, as required. \square

Lemma 2.8. *Let \dot{R} be an irreducible locally finite root system of type D and B a connected subset of \dot{R}^\times such that $\text{supp}(B) = I$. Then for any $i \in I$, the set $B \cup \{2\epsilon_i\}$ generates $\langle \dot{R} \rangle$.*

Proof. Fix $i \in I$ and let $B' := B \cup \{2\epsilon_i\}$. Since B is connected and $\text{supp}(B) = I$, for $j \in I$ there exist $\dot{\alpha}_0, \dot{\alpha}_n \in B$ such that $i \in \text{supp}(\dot{\alpha}_0)$, $j \in \text{supp}(\dot{\alpha}_n)$ and a chain $\dot{\alpha}_0, \dots, \dot{\alpha}_n$ in B , connecting $\dot{\alpha}_0$ to $\dot{\alpha}_n$. Now $2\epsilon_j \in \langle \dot{\alpha}_0, \dots, \dot{\alpha}_n, 2\epsilon_i \rangle \subseteq \langle B' \rangle$. Thus $2\epsilon_j \in \langle B' \rangle$ for all $j \in I$. Next, using the connectedness of B , for any $j, k \in I = \text{supp}(B)$ with $j \neq k$, we find $\dot{\alpha}, \dot{\beta} \in B$ with $j \in \text{supp}(\dot{\alpha})$, $k \in \text{supp}(\dot{\beta})$ and a chain in B as above connecting $\dot{\alpha} = \dot{\alpha}_0$ to $\dot{\beta} = \dot{\beta}_n$. It follows that either $\epsilon_j + \epsilon_k$ or $\epsilon_j - \epsilon_k$ belongs to the \mathbb{Z} -span of this chain. Thus $\pm(\epsilon_j \pm \epsilon_k) \in \langle B' \rangle$ for all $j, k \in I$, and so $\langle B' \rangle = \langle \dot{R} \rangle$. \square

Proposition 2.9. *Let \dot{R} be an irreducible locally finite root system of type D_I , $|I| \geq 4$, and $\dot{\mathcal{P}}$ be a subset of \dot{R}^\times . Then $\dot{\mathcal{P}}$ is a reflectable set (reflectable base) for \dot{R} if and only if $\dot{\mathcal{P}}$ is a coset spanning set (coset basis) for \dot{R}^\times in $2\langle \dot{R} \rangle$. Moreover, any reflectable set contains a reflectable base and any reflectable base is an integral base.*

Proof. Let $\dot{\mathcal{P}}$ be a coset spanning set for \dot{R}^\times in $2\langle \dot{R} \rangle$. Then $\dot{\mathcal{P}}$ contains a coset basis B for \dot{R}^\times in $2\langle \dot{R} \rangle$. Clearly $\text{supp}(B) = I$ and so by Lemma 2.1, B is connected. Then $S := (\mathcal{W}_B B) \cup \{0\}$ is an irreducible locally finite root system in $\langle B \rangle = \langle S \rangle$. If S is of type D , then by Lemma 2.7, $S = \dot{R}$ and so B and consequently $\dot{\mathcal{P}}$ is a reflectable set. Moreover, if $\dot{\mathcal{P}}$ is a coset basis, then as in the proof of Proposition 2.6, we conclude that $\dot{\mathcal{P}}$ is a reflectable base.

Now considering the above argument, we are done if we show that S can only be of type D . Suppose not, then the only possibility for S is to be of type \dot{A}_J for some index set J . Suppose this holds. If $\langle S \rangle = \langle \dot{R} \rangle$, then by Proposition 2.5, S is a reflectable set for \dot{R} , so $S = \mathcal{W}_S S = \dot{R}^\times$ which contradicts the fact that \dot{R} is not of type D . So we have $\langle S \rangle \subsetneq \langle \dot{R} \rangle$. Set $K := \langle S \rangle$, $\bar{K} := K/2K$, and for a subset T of K , denote by \bar{T} the image of T in \bar{K} , under the canonical map. We know that B is a coset basis for \dot{R}^\times in $2\langle \dot{R} \rangle$ and $K \subsetneq \langle \dot{R} \rangle$, therefore \bar{B} is a basis for the vector space \bar{K} . Since $K \subsetneq \langle \dot{R} \rangle$, we have $\bar{K} \subsetneq \langle \dot{R} \rangle / 2K$. Now $\bar{B} \subseteq \bar{\dot{\mathcal{P}}}$ and $\bar{\dot{\mathcal{P}}}$ spans $\langle \dot{R} \rangle / 2K$, so \bar{B} can be extended to a basis \bar{C} of $\langle \dot{R} \rangle / 2K$ such that $B \subsetneq C \subseteq \dot{\mathcal{P}}$. Let $\dot{\alpha} := r\epsilon_u + s\epsilon_v \in C \setminus B$, for some $r, s \in \{\pm 1\}$. Then $B \cup \{\dot{\alpha}\}$ is \mathbb{Z} -linearly independent, in particular $\dot{\alpha} \notin K = \langle B \rangle$. Now we note that S is a connected subsystem of \dot{R} and $\text{supp}(S) = I$. So using a chain in B connecting two elements whose supports contain u and v , we conclude that either $\dot{\alpha} = r\epsilon_u + s\epsilon_v \in S \subseteq K$ or $\dot{\beta} := r\epsilon_u - s\epsilon_v \in S \subseteq K$. This gives that $\dot{\beta} \in K$ as $\dot{\alpha} \notin K$. Now $\{\dot{\alpha}\} \cup B$ is linearly independent and $\dot{\beta} \in K$, thus $\{2\epsilon_u\} \cup B$ is \mathbb{Z} -linearly independent. Therefore by Lemma 2.8, $B \cup \{2\epsilon_u\}$ is a basis for $\langle \dot{R} \rangle$. So we may define a homomorphism $f : \langle \dot{R} \rangle \rightarrow \mathbb{Z}_2$ such that $f(2\epsilon_u) = 1$ and $f(B) = 0$. Since

B is a coset basis for \dot{R}^\times in $2\langle\dot{R}\rangle$ we have $2\epsilon_u \in \langle B \rangle + 2\langle\dot{R}\rangle$. But $f(2\epsilon_u) = 1$ and $f(\langle B \rangle + 2\langle\dot{R}\rangle) = \{0\}$, a contradiction.

Conversely, assume that $\dot{\mathcal{P}}$ is a reflectable set (reflectable base) for \dot{R} . Then $\langle\dot{\mathcal{P}}\rangle = \langle\dot{R}\rangle$ and so $\dot{\mathcal{P}}$ is a coset spanning set for \dot{R}^\times in $2\langle\dot{R}\rangle$. If $\dot{\mathcal{P}}$ is a reflectable base and $\dot{\mathcal{P}}$ is not a coset basis, then $\dot{\mathcal{P}}$ contains a proper subset B which is a coset basis for \dot{R}^\times in $2\langle\dot{R}\rangle$. But by what we have seen above, B is a reflectable base. This contradicts the minimality of $\dot{\mathcal{P}}$.

The last assertion in the statement follows from our above arguments together with Lemma 1.17. \square

Type G_2 : Let \dot{R} be a finite root system of type G_2 .

Proposition 2.10. *A subset $\dot{\Pi}$ of \dot{R}^\times is a reflectable base for \dot{R} if and only if $\dot{\Pi} = \{\dot{\alpha}, \dot{\beta}\}$ where $\dot{\alpha}$ is a short root, $\dot{\beta}$ is a long root and $(\dot{\alpha}, \dot{\beta}) \neq 0$.*

Proof. Assume first that $\dot{\Pi}$ is a reflectable base for \dot{R} . From Lemma 1.24, we know that $\dot{\Pi}$ contains a subset $\dot{\mathcal{M}}$ such that $\dot{\mathcal{M}}_{sh} \subseteq \dot{R}_{sh}$ is a coset basis for \dot{R}_{sh} in $\langle\dot{R}_{lg}\rangle$ and $\dot{\mathcal{M}}_{lg} \subseteq \dot{R}_{lg}$ is a coset basis for \dot{R}_{lg} in $\langle 3\dot{R}_{sh} \rangle$. But one knows that both $\langle\dot{R}_{sh}\rangle/\langle\dot{R}_{lg}\rangle$ and $\langle\dot{R}_{lg}\rangle/\langle 3\dot{R}_{sh} \rangle$ are one dimensional \mathbb{Z}_3 -vector spaces. So for any roots $\dot{\alpha} \in \dot{\Pi}_{sh}$ and $\dot{\beta} \in \dot{\Pi}_{lg}$, we may assume $\dot{\mathcal{M}}_{sh} = \{\dot{\alpha}\}$ and $\dot{\mathcal{M}}_{lg} = \{\dot{\beta}\}$. Since $\dot{\Pi}$ is connected, by Lemma 1.20, it contains roots of different lengths which are not orthogonal. So we may also assume that $(\dot{\alpha}, \dot{\beta}) \neq 0$. Then the Coxeter graph associated to $\dot{\mathcal{M}}$ is a G_2 Coxeter graph and so $\dot{\mathcal{M}}$ is a reflectable set. By minimality of $\dot{\Pi}$, we get $\dot{\mathcal{M}} = \dot{\Pi}$.

Conversely, assume that $\dot{\alpha} \in \dot{R}_{sh}$ and $\dot{\beta} \in \dot{R}_{lg}$ are such that $(\dot{\alpha}, \dot{\beta}) \neq 0$. Then the Coxeter graph associated to $\dot{\Pi} := \{\dot{\alpha}, \dot{\beta}\}$ is a G_2 Coxeter graph and so $\dot{\Pi}$ is a reflectable set. Also it follows from the fact that $\dot{\Pi}$ is of cardinality 2 that $\dot{\Pi}$ is a minimal reflectable set, in other words, a reflectable base. \square

Type F_4 : Let $\{\epsilon_1, \dots, \epsilon_4\}$ be the standard basis for \mathbb{Q}^4 and set $\dot{R} := \{0\} \cup \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i \neq j \leq 4\} \cup \{(1/2)(r_1\epsilon_1 + \dots + r_4\epsilon_4) \mid r_1, \dots, r_4 \in \{\pm 1\}\}$ which is a locally finite root system of type F_4 .

Lemma 2.11. (i) *Let $r_1, r_2, r_3, r_4, s_1, s_2, s_3, s_4 \in \{\pm 1\}$ and set $\dot{\gamma}_1 := (1/2)(r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3 + r_4\epsilon_4)$, $\dot{\gamma}_2 := (1/2)(s_1\epsilon_1 + s_2\epsilon_2 + s_3\epsilon_3 + s_4\epsilon_4)$, then $(\dot{\gamma}_1, \dot{\gamma}_2) = 0$ if and only if $|\{i \in \{1, 2, 3, 4\} \mid r_i = s_i\}| = 2$. Moreover, $\dot{\gamma}_1 + \langle\dot{R}_{lg}\rangle = \dot{\gamma}_2 + \langle\dot{R}_{lg}\rangle$ if and only if either $(\dot{\gamma}_1, \dot{\gamma}_2) = 0$ or $\dot{\gamma}_1 = \pm\dot{\gamma}_2$.*

(ii) *If $\dot{\gamma}_1, \dot{\gamma}_2 \in \dot{R}_{lg}$, then $\dot{\gamma}_1 + 2\langle\dot{R}_{sh}\rangle = \dot{\gamma}_2 + 2\langle\dot{R}_{sh}\rangle$ if and only if either $(\dot{\gamma}_1, \dot{\gamma}_2) = 0$ or $\dot{\gamma}_1 = \pm\dot{\gamma}_2$.*

Proof. (i) It is an easy verification.

(ii) The implication \Leftarrow is immediate. To see the reverse implication, let $\dot{\gamma}_1, \dot{\gamma}_2 \in \dot{R}_{lg}$ with $(\dot{\gamma}_1, \dot{\gamma}_2) \neq 0$ and $\dot{\gamma}_1 \neq \pm\dot{\gamma}_2$. Then $|\text{supp}(\dot{\gamma}_1) \cap \text{supp}(\dot{\gamma}_2)| = 1$. By symmetry on indices, we may assume $\dot{\gamma}_1 = r\epsilon_2 + s\epsilon_3$ and $\dot{\gamma}_2 = r'\epsilon_3 + s'\epsilon_4$, where $r, r', s, s' \in \{\pm 1\}$.

Now if $\dot{\gamma}_1 - \dot{\gamma}_2 \in 2\langle \dot{R}_{sh} \rangle$, then we have $r\epsilon_2 + (s - r')\epsilon_3 - s'\epsilon_4 \in \mathbb{Z}(\epsilon_1 + \dots + \epsilon_4) + 2\sum_{i=2}^4 \mathbb{Z}\epsilon_i$, which is absurd. \square

Proposition 2.12. (i) Let $\dot{\mathcal{M}}_1 \subseteq \dot{R}_{lg}$ be a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and $\dot{\mathcal{M}}_2 \subseteq \dot{R}_{sh}$ be a coset spanning set for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$, then $\dot{\mathcal{M}}_1$ and $\dot{\mathcal{M}}_2$ are connected. Also $\dot{\mathcal{M}}_2$ consists of at least two nonorthogonal elements $\dot{\alpha}_1, \dot{\alpha}_2$ with $\text{supp}(\{\dot{\alpha}_1, \dot{\alpha}_2\}) = 4$.

(ii) If $\dot{\mathcal{M}}_1 \subseteq \dot{R}_{lg}$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and $\dot{\mathcal{M}}_2 \subseteq \dot{R}_{sh}$ is a coset spanning set for \dot{R}_{sh} in $2\langle \dot{R}_{lg} \rangle$ such that $\mathcal{P} := \dot{\mathcal{M}}_1 \cup \dot{\mathcal{M}}_2$ is connected, then \mathcal{P} is a reflectable set for \dot{R} . Moreover, if $\dot{\mathcal{M}}_1$ and $\dot{\mathcal{M}}_2$ are coset bases for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$, respectively, then \mathcal{P} is a reflectable base. Furthermore, all reflectable sets and reflectable bases give rise in this manner.

Proof. (i) We first note that the \mathbb{Z}_2 -vector spaces $\langle \dot{R}_{lg} \rangle / 2\langle \dot{R}_{sh} \rangle$ and $\langle \dot{R}_{sh} \rangle / \langle \dot{R}_{lg} \rangle$ are both of dimension 2. Since $\dot{\mathcal{M}}_1$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$, $\dot{\mathcal{M}}_1$ contains a coset basis $\mathcal{B} := \{\dot{\beta}_1, \dot{\beta}_2\}$. By Lemma 2.11(ii), $(\dot{\beta}_1, \dot{\beta}_2) \neq 0$ and $\dot{\beta}_1 \neq \pm \dot{\beta}_2$. This implies that $\text{supp}(\{\dot{\beta}_1, \dot{\beta}_2\})$ is of cardinality 3. Now suppose that $\dot{\beta} \in \dot{\mathcal{M}}_1$, then as $|\text{supp}(\dot{\mathcal{M}}_1)| \leq 4$, there is $i \in \{1, 2\}$ such that $|\text{supp}(\dot{\beta}) \cap \text{supp}(\dot{\beta}_i)| = 1$. This means that any element of $\dot{\mathcal{M}}_1$ is connected to either $\dot{\beta}_1$ or $\dot{\beta}_2$. This completes the proof of connectedness of $\dot{\mathcal{M}}_1$. Also as $\dot{\mathcal{M}}_2$ is a coset spanning set for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$, $\dot{\mathcal{M}}_2$ contains a coset basis $\{\dot{\alpha}_1, \dot{\alpha}_2\}$ and so $|\text{supp}(\{\dot{\alpha}_1, \dot{\alpha}_2\})| = 4$. Without loss of generality, we assume that $\text{supp}(\dot{\alpha}_1)$ is of cardinality 4. We also mention that it follows from Lemma 2.11 that $(\dot{\alpha}_1, \dot{\alpha}_2) \neq 0$. Now we consider two cases, either $|\text{supp}(\dot{\alpha}_2)| = 1$ or $|\text{supp}(\dot{\alpha}_2)| = 4$. If $|\text{supp}(\dot{\alpha}_2)| = 1$, then for any element $\dot{\alpha}$ of $\dot{\mathcal{M}}_2$, depending on $|\text{supp}(\dot{\alpha})|$, $\dot{\alpha}$ is connected to either $\dot{\alpha}_1$ or $\dot{\alpha}_2$ and then the result follows. So suppose that $|\text{supp}(\dot{\alpha}_2)| = 4$, then any short root $\dot{\alpha}$ with $|\text{supp}(\dot{\alpha})| = 1$ is connected to both $\dot{\alpha}_1, \dot{\alpha}_2$ and any short root $\dot{\alpha}$ with $|\text{supp}(\dot{\alpha})| = 4$ is connected to either $\dot{\alpha}_1$ or $\dot{\alpha}_2$, using Lemma 2.11(i). This completes the proof.

(ii) We first note that as $\dot{\mathcal{M}}_1$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and $\langle \dot{R}_{lg} \rangle / 2\langle \dot{R}_{sh} \rangle$ is a vector space of dimension 2, $\dot{\mathcal{M}}_1$ contains at least two elements. Since \mathcal{P} and $\dot{\mathcal{M}}_1 \subseteq \dot{R}_{lg}$ are connected, one finds $\dot{\alpha}_1 \in \dot{\mathcal{M}}_2$ and $\dot{\beta}_1, \dot{\beta}_2 \in \dot{\mathcal{M}}_1$ such that $(\dot{\alpha}_1, \dot{\beta}_1) \neq 0$ and $(\dot{\beta}_1, \dot{\beta}_2) \neq 0$. Without loss of generality, we suppose that there are $r, s, m, n \in \{\pm 1\}$ such that $\dot{\beta}_1 = r\epsilon_1 + s\epsilon_2$ and $\dot{\beta}_2 = m\epsilon_1 + n\epsilon_3$. Next take $\dot{\alpha}_2 \in \dot{\mathcal{M}}_2$ to be such that $\{\dot{\alpha}_1, \dot{\alpha}_2\}$ is a coset basis for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$. We consider two cases $|\text{supp}(\dot{\alpha}_1)| = 1$ and $|\text{supp}(\dot{\alpha}_1)| = 4$. In the former case, take $\dot{S} := \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i, j \leq 3, i \neq j\}$ which is a subsystem of \dot{R} of type B_3 and note that the Coxeter graph associated to the nodes of either $\{\dot{\alpha}_1, \dot{\beta}_1, \dot{\beta}_2\}$ or $\{\dot{\alpha}_1, \dot{\beta}_2, w_{\dot{\beta}_2}(\dot{\beta}_1)\}$ is the same as the Coxeter graph of type B_3 , then it follows that $\mathcal{P}' := \{\dot{\alpha}_1\} \cup \{\dot{\beta}_1, \dot{\beta}_2\}$ is a reflectable set for \dot{S} . Therefore $\dot{S} = \mathcal{W}_{\mathcal{P}'}\mathcal{P}' \subseteq \mathcal{W}_{\mathcal{P}}\mathcal{P}$. Next, we note that as $|\text{supp}(\dot{\alpha}_1)| = 1$, we get that $|\text{supp}(\dot{\alpha}_2)| = 4$ and so $\dot{\alpha}_2 = (1/2)(r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3 + r_4\epsilon_4)$ for some $r_1, \dots, r_4 \in \{\pm 1\}$. Now as $-r_1\epsilon_1 - r_2\epsilon_2 \in \dot{S} \subseteq \mathcal{W}_{\mathcal{P}}\mathcal{P}$, we get that $\dot{\beta}_3 := r_3\epsilon_3 + r_4\epsilon_4 = w_{\dot{\alpha}_2}(-r_1\epsilon_1 - r_2\epsilon_2) \in \mathcal{W}_{\mathcal{P}}\mathcal{P}$. Now again the Coxeter graph associated to the nodes of either $\{\dot{\alpha}_1, \dot{\beta}_1, \dot{\beta}_2, \dot{\beta}_3\}$ or $\{\dot{\alpha}_1, \dot{\beta}_2, w_{\dot{\beta}_2}(\dot{\beta}_1), w_{w_{\dot{\beta}_2}(\dot{\beta}_1)}(\dot{\beta}_3)\}$ is the same as the Coxeter graph of type B_4 . This in turn together with Lemma 1.21 implies that $\mathcal{P}'' := \mathcal{P}' \cup \{\dot{\beta}_3\}$ is a reflectable set for the subsystem $\dot{R}_B := \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i, j \leq 4, i \neq j\}$. This means that

$\dot{R}_B = \mathcal{W}_{\mathcal{P}''\mathcal{P}''} \subseteq \mathcal{W}_{\mathcal{P}}\mathcal{P}$. Now to complete the proof in this case, we must show any element of $\dot{R} \setminus \{\dot{\alpha}_2\}$ whose support is of cardinality 4, belongs to $\mathcal{W}_{\mathcal{P}}\mathcal{P}$. So take $s_1, \dots, s_4 \in \{\pm 1\}$ to be such that $\dot{\beta} := (1/2)(s_1\epsilon_1 + \dots + s_4\epsilon_4) \in \dot{R} \setminus \{\dot{\alpha}_2\}$. Suppose $1 \leq t \leq 4$ and $1 \leq i_1 \leq \dots \leq i_t \leq 4$ are such that $s_{i_j} = -r_{i_j}$ for $1 \leq j \leq t$, then $\dot{\beta} = w_{\epsilon_{i_1}} \dots w_{\epsilon_{i_t}} \dot{\alpha}_2 \in \mathcal{W}_{\mathcal{P}}\mathcal{P}$. This completes the proof in the former case. In the latter case, we first note that by Lemma 2.11, $(\dot{\alpha}_1, \dot{\alpha}_2) \neq 0$. Also by Lemma 1.21, \mathcal{P} is a reflectable set if and only if $(\mathcal{P} \setminus \{\dot{\alpha}_2\}) \cup \{w_{\dot{\alpha}_1}\dot{\alpha}_2\}$ is a reflectable set, so without loss of generality we assume $\text{supp}(\dot{\alpha}_2)$ is of cardinality 1. Since $(\dot{\beta}_1, \dot{\alpha}_1) \neq 0$, we get $\dot{\alpha}_1 = \pm(1/2)(r\epsilon_1 + s\epsilon_2 + r'\epsilon_3 + s'\epsilon_4)$ for some $r', s' \in \{\pm 1\}$. Set $\dot{\beta}_3 := w_{\dot{\alpha}_1}(\dot{\beta}_1)$, then $\text{supp}(\dot{\beta}_3) = \{3, 4\}$ and so by Proposition 2.4, $\mathcal{P}_1 := \{\dot{\alpha}_2, \dot{\beta}_1, \dot{\beta}_2, \dot{\beta}_3\}$ is a reflectable set for \dot{R}_B i.e., $\{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i \neq j \leq 4\} = \mathcal{W}_{\mathcal{P}_1}\mathcal{P}_1$ and so the same argument as above completes the proof.

Now suppose $\dot{\mathcal{M}}_1$ and $\dot{\mathcal{M}}_2$ are coset bases for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$, respectively, but $\mathcal{P} = \dot{\mathcal{M}}_1 \cup \dot{\mathcal{M}}_2$ is not a reflectable base. So there is $\dot{\mathcal{P}} \subseteq \mathcal{P}$ such that $\dot{\mathcal{P}}$ is a reflectable set for \dot{R} . Thus by Lemma 1.24, $\dot{\mathcal{P}}_{sh}$ and $\dot{\mathcal{P}}_{lg}$ are coset spanning sets for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$, respectively. But one knows that $\dot{\mathcal{P}}_{lg} \subseteq \dot{\mathcal{M}}_1$ and $\dot{\mathcal{P}}_{sh} \subseteq \dot{\mathcal{M}}_2$, therefore we get $\dot{\mathcal{M}}_1 = \dot{\mathcal{P}}_{lg}$ and $\dot{\mathcal{M}}_2 = \dot{\mathcal{P}}_{sh}$ and so $\mathcal{P} = \dot{\mathcal{P}}$. It is easy to see, using Lemma 1.24, that any reflectable set is connected and that it is of the form $\dot{\mathcal{M}}_1 \cup \dot{\mathcal{M}}_2$ where $\dot{\mathcal{M}}_1$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and $\dot{\mathcal{M}}_2$ is a coset spanning set for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$. Now suppose that $\dot{\Pi}$ is a reflectable base for \dot{R} . We have already seen that $\dot{\Pi} = \dot{\Pi}_{lg} \uplus \dot{\Pi}_{sh}$, and that $\dot{\Pi}_{lg}$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and $\dot{\Pi}_{sh}$ is a coset spanning set for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$. Since $\dot{\Pi}$ is connected, there are $\dot{\alpha} \in \dot{\Pi}_{lg}$ and $\dot{\beta} \in \dot{\Pi}_{sh}$ such that $(\dot{\alpha}, \dot{\beta}) \neq 0$. Take $\dot{\gamma} \in \dot{\Pi}_{lg}$ and $\dot{\eta} \in \dot{\Pi}_{sh}$ such that $\{\dot{\alpha}, \dot{\gamma}\}$ is a coset basis for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ and $\{\dot{\beta}, \dot{\eta}\}$ is a coset basis for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$. Therefore, $\{\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\eta}\} \subseteq \dot{\Pi}$ is a reflectable base for \dot{R} and so $\dot{\Pi} = \{\dot{\beta}, \dot{\eta}, \dot{\alpha}, \dot{\gamma}\}$. This completes the proof. \square

Type C_I ($|I| \geq 3$): Let \dot{R} be a locally finite root system of type C_I ($|I| \geq 3$) and \dot{R}^\vee be its dual root system, namely $\dot{R}^\vee = \{0\} \cup \{\dot{\alpha}^\vee := 2\dot{\alpha}/(\dot{\alpha}, \dot{\alpha}) \mid \dot{\alpha} \in \dot{R}^\times\}$. It is known that \dot{R}^\vee is a locally finite root system of type B_I .

Proposition 2.13. *Let $\dot{\Pi} \subseteq \dot{R}^\times$. Then $\dot{\Pi}$ is a reflectable base for \dot{R} if and only if $\dot{\Pi} = \{\dot{\beta}\} \cup \dot{\mathcal{M}}$, where $\dot{\beta}$ is a long root and $\dot{\mathcal{M}}$ is a coset basis for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$.*

Proof. Set $\dot{\Pi}^\vee := \{\dot{\alpha}^\vee \mid \dot{\alpha} \in \dot{\Pi}\}$. Let $\dot{\alpha} \in \mathcal{W}_{\dot{\Pi}}\dot{\Pi}$, say $\dot{\alpha} = s_{\dot{\alpha}_1} \dots s_{\dot{\alpha}_n}(\dot{\alpha}_{n+1})$ where $\dot{\alpha}_i$'s are in $\dot{\Pi}$. Then $\dot{\alpha}^\vee = s_{\dot{\alpha}_1^\vee} \dots s_{\dot{\alpha}_n^\vee}(\dot{\alpha}_{n+1}^\vee)$. This shows that $(\mathcal{W}_{\dot{\Pi}}\dot{\Pi})^\vee = \mathcal{W}_{\dot{\Pi}^\vee}(\dot{\Pi}^\vee)$. From this it follows that $\dot{\Pi}$ is a reflectable base for \dot{R} if and only if $\dot{\Pi}^\vee$ is a reflectable base for \dot{R}^\vee . Moreover, it is easy to see that dual of a coset basis for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$ is a minimal coset spanning set for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$ and vice versa. Now the result follows immediately from Proposition 2.4. \square

Corollary 2.14. *Suppose that \dot{R} is a locally finite root system of a non-simply laced type. Then any reflectable set for \dot{R} contains a reflectable base for \dot{R} and any reflectable base is an integral base.*

Proof. Using Lemma 1.24, Propositions 2.3, 2.4, 2.10, 2.12 and 2.13, we get the first assertion. For the second assertion, using Propositions 2.4, 2.12, 2.10 and 2.13, we get that $\dot{\Pi}_{sh}, \dot{\Pi}_{lg}$ are coset bases for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$ and for \dot{R}_{lg} in $\langle \rho \dot{R}_{sh} \rangle$, respectively. Now as $\rho \langle \dot{R}_{sh} \rangle \subseteq \langle \dot{R}_{lg} \rangle \subseteq \langle \dot{R}_{sh} \rangle$, we are done using Lemma 1.17. \square

Remark 2.15. Suppose that \dot{R} is a locally finite root system of a non-simply laced type. By Corollary 2.14, we know that any reflectable base is an integral base. This in particular implies that $|\dot{\Pi}| = \text{rank}(\dot{R})$. Moreover, it follows from the proof of Propositions 2.3, 2.4, 2.10 and 2.12 that $|\dot{\Pi}|$ can be characterized as

$$\begin{aligned} |\dot{\Pi}| &= \dim(\langle \dot{R}_{sh} \rangle / \langle \dot{R}_{lg} \rangle) + \dim(\langle \dot{R}_{lg} \rangle / \langle \rho \dot{R}_{sh} \rangle) \\ &= \begin{cases} 1 + \dim(\langle \dot{R}_{lg} \rangle / \langle 2\dot{R}_{sh} \rangle) & \text{if } X = B_I, \\ \dim(\langle \dot{R}_{sh} \rangle / \langle \dot{R}_{lg} \rangle) + 1 & \text{if } X = C_I, \\ 4 & \text{if } X = F_4, \\ 2 & \text{if } X = G_2. \end{cases} \end{aligned}$$

We conclude this section with the following two propositions which will not be used in the sequel but reveal some interesting relations between root systems of types A_I , B_I and D_I . Let \dot{R} be a locally finite root system of type B_I , we recall that \dot{R} is of the form $\{0, \pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid i \neq j \in I\}$. Take $S_A := \{\pm(\epsilon_i - \epsilon_j) \mid i, j \in I, i \neq j\} \cup \{0\}$ and $S_D := \{\pm(\epsilon_i \pm \epsilon_j) \mid i, j \in I, i \neq j\} \cup \{0\}$. Then S_A is a locally finite root system of type A_I and S_D is a locally finite root system of type D_I (if $|I| \geq 4$).

Proposition 2.16. (i) Let \dot{R} be a locally finite root system of type B_I and S_A be the subsystem of \dot{R} of type A_I as above. Suppose that $\dot{\mathcal{P}} \subseteq S_A^\times$ and fix $i \in I$. Then $\dot{\mathcal{P}}$ is a reflectable set for S_A if and only if $\dot{\Pi} := \dot{\mathcal{P}} \cup \{\epsilon_i\}$ is a reflectable set for \dot{R} . Moreover, $\dot{\mathcal{P}}$ is a reflectable base for S_A if and only if $\dot{\Pi}$ is a reflectable base for \dot{R} .

Proof. (i) Suppose that $\dot{\mathcal{P}}$ is a reflectable set for S_A . Then $\dot{\mathcal{P}}$ is connected and $\langle \dot{\mathcal{P}} \rangle = \langle S_A \rangle$. So $\dot{\mathcal{P}}$ is a connected coset spanning set for \dot{R}_{lg} in $\langle 2\dot{R}_{sh} \rangle$. Therefore by Proposition 2.4(iii), $\dot{\Pi}$ is a reflectable set for \dot{R} . Next suppose that $\dot{\Pi}$ is a reflectable set for \dot{R} . We show that $S_A^\times = \mathcal{W}_{\dot{\mathcal{P}}} \dot{\mathcal{P}}$. Suppose that $r, s \in I$ with $r \neq s$. We prove that $\epsilon_r - \epsilon_s \in \mathcal{W}_{\dot{\mathcal{P}}} \dot{\mathcal{P}}$. Since $\dot{\Pi}$ is a reflectable set for \dot{R} , there are $\dot{\alpha}_1, \dots, \dot{\alpha}_n \in \dot{\Pi}$ and $\dot{\alpha} \in \dot{\mathcal{P}}$ such that $w_{\dot{\alpha}_1} \dots w_{\dot{\alpha}_n}(\dot{\alpha}) = \epsilon_r - \epsilon_s$, with n as small as possible (we refer to $w_{\dot{\alpha}_1} \dots w_{\dot{\alpha}_n}(\dot{\alpha})$ as a “reduce expression”). We show that $w_{\dot{\alpha}_1} \dots w_{\dot{\alpha}_n}(\dot{\alpha}) \in \mathcal{W}_{\dot{\mathcal{P}}} \dot{\mathcal{P}}$. Let $f : \dot{A} \rightarrow \mathbb{Z}$ be the homomorphism induced by the assignment $\epsilon_j \mapsto 1$ for all $j \in I$. Now $0 = f(\epsilon_r - \epsilon_s) = f(w_{\dot{\alpha}_1} \dots w_{\dot{\alpha}_n}(\dot{\alpha}))$. Note that if $\dot{\alpha}_k = \epsilon_i$ for some $1 \leq k \leq n$, then $(\dot{\alpha}_k, w_{\dot{\alpha}_{k+1}} \dots w_{\dot{\alpha}_n}(\dot{\alpha})) \neq 0$ as the expression is reduced and $f(\dot{\alpha}_j) = 0$ for $\dot{\alpha}_j \in \dot{\mathcal{P}}$. Thus $0 = f(w_{\dot{\alpha}_1} \dots w_{\dot{\alpha}_n}(\dot{\alpha})) = \sum_{t=1}^p k_t$, where $k_t \in \{\pm 2\}$ and p is the number of j 's for which $\dot{\alpha}_j = \epsilon_i$. Therefore p is even and so without loss of generality, we may assume that $\dot{\alpha}_1 = \dot{\alpha}_n = \epsilon_i$, and $\dot{\alpha}_j \neq \epsilon_i$ for $2 \leq j \leq n-1$. Now as the expression is reduced, we have $(\epsilon_i, \epsilon_r - \epsilon_s) = (\dot{\alpha}, \epsilon_r - \epsilon_s) \neq 0$ and $(\epsilon_i, \dot{\alpha}) = (\dot{\alpha}_n, \dot{\alpha}) \neq 0$. Now one can see that $w_{\dot{\alpha}_1} w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_n}(\dot{\alpha}), w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_{n-1}}(\dot{\alpha}) \in S_A^\times$ and $\text{supp}(w_{\dot{\alpha}_1} w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_n}(\dot{\alpha})) = \text{supp}(w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_{n-1}}(\dot{\alpha}))$. Now the minimality of n implies

that $w_{\dot{\alpha}_1} w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_n}(\dot{\alpha}) = -w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_{n-1}}(\dot{\alpha})$. Setting $\dot{\beta}_1 := w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_{n-1}}(\dot{\alpha})$, we have $w_{\dot{\alpha}_1} w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_n}(\dot{\alpha}) = w_{\dot{\beta}} w_{\dot{\alpha}_2} \dots w_{\dot{\alpha}_{n-1}}(\dot{\alpha}) \in \mathcal{W}_{\dot{\mathcal{P}}} \dot{\mathcal{P}}$. This completes the proof of the first assertion. The second assertion immediately follows from the first one. \square

Proposition 2.17. *Let \dot{R} be a locally finite root system of type B_I and $S_D \subseteq \dot{R}$ be the subsystem of type D_I introduced above.*

(i) *Suppose that $\dot{\mathcal{P}}$ is a reflectable set for S_D and there exist $\dot{\alpha}, \dot{\beta} \in \dot{\mathcal{P}}$ with $\text{supp}(\dot{\alpha}) = \text{supp}(\dot{\beta}) = \{i, j\}$ and $(\dot{\alpha}, \dot{\beta}) = 0$. Then $\dot{\Pi} := (\dot{\mathcal{P}} \setminus \{\dot{\alpha}\}) \cup \{\epsilon_i\}$ is a reflectable set for \dot{R} .*

(ii) *Suppose $\dot{\mathcal{P}} \subseteq S_D$ and $i \in I$ are such that $\dot{\Pi} := \dot{\mathcal{P}} \cup \{\epsilon_i\}$ is a reflectable set for \dot{R} . Let $\dot{\alpha} = r\epsilon_i + s\epsilon_j \in \dot{\mathcal{P}}$ and set $\dot{\beta} := r\epsilon_i - s\epsilon_j$. Then $\dot{\Pi}' := \dot{\mathcal{P}} \cup \{\dot{\beta}\}$ is a reflectable set for S_D .*

Proof. (i) By assumption, we have $\dot{\alpha} = r\epsilon_i + s\epsilon_j$ and $\dot{\beta} = r\epsilon_i - s\epsilon_j$ for some $i, j \in I$ and $r, s \in \{\pm 1\}$. Now as $\dot{\beta}, \epsilon_i \in \dot{\Pi}$, we have $-\dot{\alpha} = -r\epsilon_i - s\epsilon_j = w_{\epsilon_i}(\dot{\beta}) \in \mathcal{W}_{\dot{\Pi}} \dot{\Pi}$. This gives that $\dot{\mathcal{P}} \subseteq \mathcal{W}_{\dot{\Pi}} \dot{\Pi}$. Now as $\dot{\mathcal{P}}$ is a reflectable set for S_D , $S_D \subseteq \mathcal{W}_{\dot{\Pi}} \dot{\Pi}$. Then for any $k \in I$, $\epsilon_k = w_{\epsilon_i - \epsilon_k} \epsilon_i \in \mathcal{W}_{\dot{\Pi}} \dot{\Pi}$. Thus $\mathcal{W}_{\dot{\Pi}} \dot{\Pi}$ contains all nonzero roots of \dot{R} and so $\dot{\Pi}$ is a reflectable set as required.

(ii) We first note that if $k \in I$ and $\dot{\gamma} \in S_D^\times$, then

$$\begin{cases} w_{\epsilon_i}(\dot{\gamma}) = w_{\dot{\alpha}} w_{\dot{\beta}}(\dot{\gamma}) & \text{if } i \in \text{supp}(\dot{\gamma}), \dot{\gamma} \neq \pm \dot{\alpha}, \pm \dot{\beta}, \\ w_{\epsilon_i}(\dot{\gamma}) = \dot{\gamma} & \text{if } i \notin \text{supp}(\dot{\gamma}), \\ w_{\epsilon_i}(\dot{\gamma}) = \pm \dot{\beta}, & \text{if } \dot{\gamma} = \pm \dot{\alpha}, \\ w_{\epsilon_i}(\dot{\gamma}) = \pm \dot{\alpha}, & \text{if } \dot{\gamma} = \pm \dot{\beta}. \end{cases} \quad (2.18)$$

We must show that $S_D^\times \subseteq \mathcal{W}_{\dot{\Pi}'} \dot{\Pi}'$. Take $\eta \in S_D^\times$. Since $\dot{\Pi}$ is a reflectable set for \dot{R} , there are $\dot{\alpha}_1, \dots, \dot{\alpha}_t, \dot{\alpha}' \in \dot{\Pi}$ such that $w_{\dot{\alpha}_1} \dots w_{\dot{\alpha}_t}(\dot{\alpha}') = \eta$ which in turn implies that $\dot{\alpha}' \in \dot{\mathcal{P}}$. If for all $1 \leq k \leq t$, $\dot{\alpha}_k \neq \epsilon_i$, there is nothing to prove. Otherwise, suppose $1 \leq k \leq t$ is such that $\dot{\alpha}_k = \epsilon_i$ and $\dot{\alpha}_s \neq \epsilon_i$ for $k+1 \leq s \leq t$. Using (2.18), we can replace $w_{\dot{\alpha}_k} \dots w_{\dot{\alpha}_t}(\dot{\alpha}')$ with an element of $\mathcal{W}_{\dot{\Pi}'} \dot{\Pi}'$. We do the same for other reflections based on ϵ_i appearing in the expression and so we get that $\eta \in \mathcal{W}_{\dot{\Pi}'} \dot{\Pi}'$. \square

3. Characterization of reflectable bases

In this section, we give a full characterization of reflectable sets and reflectable bases for tame irreducible reflection systems of reduced types $X \neq E_{6,7,8}$. We recall the definition of a coset spanning set and a strong coset spanning set from Section 1. Throughout this section, $(A, (\cdot, \cdot), R)$ is a tame irreducible affine reflection system and \mathcal{W} is its Weyl group. We recall from Section 1 that $\bar{A} = A/A^0$ and that $\bar{\cdot} : A \rightarrow \bar{A}$ is the canonical epimorphism. We also recall the map \mathbf{p} as in (1.18).

Type A_1 : Considering Theorem 1.13, we get that in this case $R = (S + S) \cup (\dot{R}^\times + S)$ in which $\dot{R} = \{\pm \dot{\alpha}\} \cup \{0\}$ is a finite root system of type A_1 and S is a pointed reflection subspace of A^0 .

Theorem 3.1. *Suppose $\Pi \subseteq R^\times$ with $\langle \Pi \rangle = \langle R \rangle$. Then Π is a reflectable set (resp. a reflectable base) for R if and only if Π is a strong coset spanning set (resp. a minimal strong coset spanning set) for R^\times in $2\langle R \rangle$ with respect to $\langle R \rangle$.*

Proof. Let Π be a reflectable set. Then $\langle \Pi \rangle = \langle \mathcal{W}_\Pi \Pi \rangle = \langle R \rangle$. Moreover, one can easily see that

$$\mathcal{W}_\Pi \alpha \subseteq \alpha + 2\langle \Pi \rangle \quad (\alpha \in \Pi).$$

Therefore,

$$R^\times = \cup_{\alpha \in \Pi} \mathcal{W}_\Pi \alpha \subseteq \cup_{\alpha \in \Pi} [(\alpha + 2\langle \Pi \rangle) \cap R^\times] = \cup_{\alpha \in \Pi} [(\alpha + 2\langle R \rangle) \cap R^\times] \subseteq R^\times.$$

This means that Π is a strong coset spanning set for R^\times in $2\langle R \rangle$ with respect to $\langle R \rangle$.

Next assume Π is a strong coset spanning set for R^\times in $2\langle R \rangle$ with respect to $\langle R \rangle$. Then

$$R^\times = \cup_{\alpha \in \Pi} [(\alpha + 2\langle R \rangle) \cap R^\times]. \quad (3.2)$$

Since $R^\times = \pm \dot{\alpha} + S$, using the sign freeness of Π , we can (and do) assume that $\beta - \mathbf{p}(\beta) = \dot{\alpha}$ for all $\beta \in \Pi$. Since $\dot{\alpha} \in R^\times$ and $\langle \Pi \rangle = \langle R \rangle = \mathbb{Z}\dot{\alpha} + \langle S \rangle$, it follows from (3.2) that there exists $\delta \in \langle S \rangle$ such that $\alpha := \dot{\alpha} + 2\delta \in \Pi$. Since Π is a strong coset spanning set for R^\times in $2\langle R \rangle$, it can be seen that $\Pi_\alpha := \Pi - \alpha$ satisfies

$$\langle \Pi_\alpha \rangle = \langle S \rangle \quad \text{and} \quad S = \cup_{\sigma \in \Pi_\alpha} (\sigma + 2\langle S \rangle). \quad (3.3)$$

Next, we note that for $\zeta \in \Pi_\alpha$, $-\alpha + \zeta = w_\alpha(\alpha + \zeta)$ which in turn implies that (by sign freeness)

$$\begin{aligned} r\alpha + \zeta &\in \mathcal{W}_\Pi \Pi \quad \text{and} \quad w_{\alpha+r\zeta} \in \mathcal{W}_\Pi; \\ r &\in \{\pm 1\}, \zeta \in \Pi_\alpha. \end{aligned} \quad (3.4)$$

Also one can easily see that for $r, r_1, \dots, r_n \in \{\pm 1\}$ and $\zeta, \zeta_1, \dots, \zeta_n \in \Pi_\alpha$, we have

$$w_{\alpha+r_n\zeta_n} \dots w_{\alpha+r_1\zeta_1} (r\alpha + \zeta) = (-1)^n r\alpha + \zeta - 2rr_1\zeta_1 + 2rr_2\zeta_2 + \dots + 2(-1)^n rr_n\zeta_n.$$

Now this together with (3.4) and (3.3) implies that

$$\begin{aligned} \mathcal{W}_\Pi \Pi &= \pm\alpha + \Pi_\alpha + 2\langle \Pi_\alpha \rangle \\ (\text{by (3.3)}) &= \pm\alpha + \Pi_\alpha + \langle 2S \rangle \\ &= \pm\alpha + S \\ &= R^\times \end{aligned}$$

which means that Π is a reflectable set for R . This completes the proof. \square

Type B:

Lemma 3.5. *Suppose that $\Pi \subseteq R^\times$ is a reflectable set for R , then Π_{sh} is a strong coset spanning set for R_{sh} in $\langle R_{lg} \rangle$ with respect to $\langle R \rangle$ and Π_{lg} is a coset spanning set for R_{lg} in $2\langle R_{sh} \rangle$. Moreover if $|I| = 2$, then Π_{lg} is a strong coset spanning set for R_{lg} in $2\langle R_{sh} \rangle$ with respect to $\langle R \rangle$. In other words,*

$$R_{sh} = \bigcup_{\alpha \in \Pi_{sh}} [(\alpha + \langle R_{lg} \rangle) \cap R_{sh}],$$

$$R_{lg} = \bigcup_{\alpha \in \Pi_{lg}} [(\alpha + 2\langle R_{sh} \rangle) \cap R_{lg}], \quad (X = B_2)$$

and

$$\langle R_{lg} \rangle = \langle \Pi_{lg} \rangle + \langle 2R_{sh} \rangle.$$

Proof. Using relations stated in Proposition 1.13 and the well known facts about locally finite root systems, we have

$$\langle 2R_{sh} \rangle + \langle R_{lg} \rangle \subseteq \langle R_{lg} \rangle \quad \text{and} \quad \langle R_{lg} \rangle + \langle R_{sh} \rangle \subseteq \langle R_{sh} \rangle. \quad (3.6)$$

Now considering the different possibilities for (α, β^\vee) , $\alpha, \beta \in R^\times$ and using (3.6), one can check that for $w \in \mathcal{W}$,

$$w(\alpha) \in \alpha + \langle R_{lg} \rangle; \quad (\alpha \in R_{sh}, w \in \mathcal{W}) \quad (3.7)$$

and

$$w(\beta) \in \beta + 2\langle R_{sh} \rangle; \quad (X = B_2, w \in \mathcal{W}, \beta \in R_{lg}). \quad (3.8)$$

Moreover, we note if $\alpha_1, \dots, \alpha_m \in R^\times$ and $\{\beta_1, \dots, \beta_t\} = \{\alpha_1, \dots, \alpha_m\} \cap R_{lg}$, then for $\alpha \in R_{lg}$, we have

$$w_{\alpha_1} \cdots w_{\alpha_m}(\alpha) + 2\langle R_{sh} \rangle = w_{\beta_1} \cdots w_{\beta_t}(\alpha) + 2\langle R_{sh} \rangle.$$

Therefore, $R_{lg} = \mathcal{W}\Pi_{lg} \subseteq \mathcal{W}\Pi_{lg}\Pi_{lg} + 2\langle R_{sh} \rangle$ and so

$$\langle R_{lg} \rangle \subseteq \langle \Pi_{lg} \rangle + 2\langle R_{sh} \rangle \subseteq \langle R_{lg} \rangle. \quad (3.9)$$

Now from (3.7), (3.8), (3.9) and $\mathcal{W}\Pi = R^\times$, we have

$$R_{sh} = \mathcal{W}\Pi_{sh} = \bigcup_{w \in \mathcal{W}, \alpha \in \Pi_{sh}} w(\alpha) \subseteq \bigcup_{\alpha \in \Pi_{sh}} [(\alpha + \langle R_{lg} \rangle) \cap R_{sh}] \subseteq R_{sh},$$

$$R_{lg} = \mathcal{W}\Pi_{lg} = \bigcup_{w \in \mathcal{W}, \alpha \in \Pi_{lg}} w(\alpha) \subseteq \bigcup_{\alpha \in \Pi_{lg}} [(\alpha + 2\langle R_{sh} \rangle) \cap R_{lg}] \subseteq R_{lg} \quad (X = B_2),$$

and

$$\langle R_{lg} \rangle = \langle \Pi_{lg} \rangle + 2\langle R_{sh} \rangle.$$

Therefore the equalities in the statement hold. This completes the proof. \square

Type B_2 :

Lemma 3.10. *Consider a description*

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \quad (3.11)$$

for R as in Theorem 1.13. Let $\mathcal{R}_1 := \{\sigma_i \mid i \in J\}$ be a strong coset spanning set for S in $\langle L \rangle$ and $\mathcal{R}_2 := \{\tau_j \mid j \in K\}$ be a strong coset spanning set for L in $\langle 2S \rangle$, such that

$$\langle \mathcal{R}_1 \rangle + \langle \mathcal{R}_2 \rangle = \langle S \rangle. \quad (3.12)$$

Assume that $0 \in \mathcal{R}_1$ and $0 \in \mathcal{R}_2$. For $i \in J$ and $j \in K$, pick $\dot{\alpha}_i \in \dot{R}_{sh}$ and $\dot{\beta}_j \in \dot{R}_{lg}$, and set $\Pi_{sh} := \{\dot{\alpha}_i + \sigma_i \mid i \in J\}$ and $\Pi_{lg} := \{\dot{\beta}_j + \tau_j \mid j \in K\}$. Then $\Pi := \Pi_{sh} \cup \Pi_{lg}$ is a reflectable set for R .

Proof. Since $\{\sigma_i \mid i \in J\}$ is a strong coset spanning set for S in $\langle L \rangle$, we have by definition that $S = \cup_{i \in J}[(\sigma_i + \langle L \rangle) \cap S]$ and $L = \cup_{j \in K}[(\tau_j + \langle 2S \rangle) \cap L]$. However, $S + L \subseteq S$ and $2S + L \subseteq L$, so using (3.12), we have

$$S = \cup_{i \in J}(\sigma_i + \langle L \rangle), \quad L = \cup_{j \in K}(\tau_j + \langle 2S \rangle) \quad \text{and} \quad 2\langle \mathcal{R}_1 \rangle + \langle \mathcal{R}_2 \rangle = \langle L \rangle. \quad (3.13)$$

We must show that $\mathcal{W}_\Pi \Pi = R^\times$. Since $0 \in \mathcal{R}_1$ and $0 \in \mathcal{R}_2$, we have $\dot{R}_{sh} \cap \Pi \neq \emptyset$ and $\dot{R}_{lg} \cap \Pi \neq \emptyset$. Then it is easy to see that $\dot{R}^\times \subseteq \mathcal{W}_\Pi \Pi$ and so $\dot{W} \subseteq \mathcal{W}_\Pi$. Therefore for $j \in J$ and $k \in K$,

$$\dot{R}_{sh} \pm \sigma_j \subseteq \dot{W}(\dot{\alpha}_j \pm \sigma_j) \subseteq \mathcal{W}_\Pi \Pi \quad \text{and} \quad \mathcal{W}_{\dot{R}_{sh} \pm \sigma_j} \subseteq \mathcal{W}_\Pi,$$

and

$$\dot{R}_{lg} \pm \tau_k \subseteq \dot{W}(\dot{\beta}_k \pm \tau_k) \subseteq \mathcal{W}_\Pi \Pi \quad \text{and} \quad \mathcal{W}_{\dot{R}_{lg} \pm \tau_k} \subseteq \mathcal{W}_\Pi.$$

Next for $i, j \in J$ and $k, t \in K$, we have

$$\dot{R}_{sh} \pm \sigma_j \pm \tau_k \pm 2\sigma_i \subseteq \dot{W}\mathcal{W}_{\dot{R}_{sh} \pm \sigma_i} \mathcal{W}_{\dot{R}_{lg} \pm \tau_k} (\dot{R}_{sh} \pm \sigma_j) \subseteq \mathcal{W}_\Pi \Pi,$$

and

$$\dot{R}_{lg} \pm \tau_k \pm 2\sigma_j \pm 2\tau_t \subseteq \dot{W}\mathcal{W}_{\dot{R}_{lg} \pm \tau_t} \mathcal{W}_{\dot{R}_{sh} \pm \sigma_j} (\dot{R}_{lg} \pm \tau_k) \subseteq \mathcal{W}_\Pi \Pi.$$

For a fixed $j \in J$, repeating this argument, we have

$$\dot{R}_{sh} \pm \sigma_j + \langle \tau_k \mid k \in K \rangle + 2\langle \sigma_i \mid i \in J \rangle \subseteq \mathcal{W}_\Pi \Pi,$$

and

$$\dot{R}_{lg} \pm \tau_k + 2\langle \tau_t \mid t \in K \rangle + 2\langle \sigma_i \mid i \in J \rangle \subseteq \mathcal{W}_\Pi \Pi.$$

Now using (3.13) and (3.12), we have

$$\dot{R}_{sh} + \sigma_j + \langle L \rangle = \dot{R}_{sh} + \sigma_j + \langle \mathcal{R}_2 \rangle + 2\langle \mathcal{R}_1 \rangle \subseteq \mathcal{W}_\Pi \Pi,$$

and

$$\dot{R}_{lg} + \tau_k + 2\langle S \rangle = \dot{R}_{lg} + \tau_k + 2\langle \mathcal{R}_2 \rangle + 2\langle \mathcal{R}_1 \rangle \subseteq \mathcal{W}_\Pi \Pi.$$

Since $j \in J$ and $k \in K$ were chosen arbitrary, we get from (3.13) that

$$\dot{R}_{sh} + S \subseteq \mathcal{W}_\Pi \Pi \quad \text{and} \quad \dot{R}_{lg} + L \subseteq \mathcal{W}_\Pi \Pi.$$

This completes the proof that Π is a reflectable set. \square

Theorem 3.14. *Suppose that Π is a subset of R^\times with $\langle \Pi \rangle = \langle R \rangle$. Then Π is a reflectable set (resp. reflectable base) for R if and only if $\Pi_{sh} = \Pi \cap R_{sh}$ is a strong coset spanning set (resp. a minimal strong coset spanning set) for R_{sh} in $\langle R_{lg} \rangle$ and $\Pi_{lg} = \Pi \cap R_{lg}$ is a strong coset spanning set (resp. a minimal strong coset spanning set) for R_{lg} in $2\langle R_{sh} \rangle$, with respect to $\langle R \rangle$.*

Proof. By Lemma 3.5, it is enough to show that if Π is a subset of R^\times satisfying

- $\langle \Pi \rangle = \langle R \rangle$,
- Π_{sh} is a strong coset spanning set for R_{sh} in $\langle R_{lg} \rangle$ with respect to $\langle R \rangle$,
- Π_{lg} is a strong coset spanning set for R_{lg} in $2\langle R_{sh} \rangle$ with respect to $\langle R \rangle$,

then it is a reflectable set. So suppose $\Pi \subseteq R^\times$ satisfies the conditions mentioned above. Therefore $\Pi = \mathcal{R}'_1 \cup \mathcal{R}'_2$, where \mathcal{R}'_1 is a strong coset spanning set for R_{sh} in $\langle R_{lg} \rangle$ and \mathcal{R}'_2 is a strong coset spanning set for R_{lg} in $2\langle R_{sh} \rangle$. Note that $\bar{\Pi}$ contains at least one short root and one long root and so is a reflectable set for \bar{R} by Proposition 2.3. Therefore by Proposition 2.14, it contains an integral reflectable base \mathcal{P} . Take a pre-image $\dot{\Pi} \subseteq \Pi$ of \mathcal{P} and construct a finite root system of type B_2 , denoted again by \dot{R} , and (S, L) as usual to get the description (3.11) for R (see the proof of Theorem 1.13

for details). Since $\dot{\Pi} \subseteq \Pi \cap \dot{R}$, we have $\dot{R} \cap \Pi_{sh} \neq \emptyset$ and $\dot{R} \cap \Pi_{lg} \neq \emptyset$. Let $\dot{\alpha} \in \dot{R} \cap \Pi_{sh}$, and assume $\dot{\beta} \in \dot{R} \cap \Pi_{lg}$, $\Pi_{sh} = \{\alpha_j \mid j \in J\}$ and $\Pi_{lg} = \{\beta_t \mid t \in T\}$. We know that $\mathfrak{p}(\alpha_j) \in S$ with $\mathfrak{p}(\dot{\alpha}) = 0$ and that $\alpha_j - \mathfrak{p}(\alpha_j) - \dot{\alpha} \in \langle \dot{R}_{lg} \rangle$ for all $j \in J$. Therefore for $j \in J$, we have

$$\alpha_j + \langle R_{lg} \rangle = \dot{\alpha} + \mathfrak{p}(\alpha_j) + \langle R_{lg} \rangle.$$

Now let $\tau \in S$, then $\dot{\alpha} + \tau \in R_{sh} = \cup_{j \in J}[(\alpha_j + \langle R_{lg} \rangle) \cap R_{sh}]$. So $\dot{\alpha} + \tau \in \alpha_j + \langle R_{lg} \rangle = \dot{\alpha} + \mathfrak{p}(\alpha_j) + \langle R_{lg} \rangle$, for some $j \in J$. This gives $\tau \in \cup_{j \in J}(\mathfrak{p}(\alpha_j) + \langle L \rangle)$. This means that $\mathcal{R}_1 := \{\mathfrak{p}(\alpha_j) \mid j \in J\}$ is a strong coset spanning set for S in $\langle L \rangle$. Using a similar argument, we see that $\mathcal{R}_2 := \{\mathfrak{p}(\beta_t) \mid t \in T\}$ is a strong coset spanning set for L in $2\langle S \rangle$. We also note that, as we have already seen, $0 \in \mathcal{R}_1$ and $0 \in \mathcal{R}_2$. Finally, since $\langle \Pi \rangle = \langle R \rangle$, we have $\langle \mathcal{R}_1 \rangle + \langle \mathcal{R}_2 \rangle = \langle S \rangle$. Therefore all conditions in the statement of Lemma 3.10 hold and so Π is a reflectable set for R . \square

Type B_I, C_I ($|I| \geq 3$): We give the proof for type B , the proof for type C is analogous, replacing the roles of short and long roots. So from now on, we assume that we are in type B . We recall that in this case L is a lattice.

Lemma 3.15. *Consider a description*

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \quad (3.16)$$

for R as in Theorem 1.13. Suppose that $\mathcal{R} := \{\sigma_j \mid j \in J\}$ is a strong coset spanning set for S in L with $0 \in \mathcal{R}$, and $\mathcal{M} := \{\tau_k \mid k \in K\}$ is a coset spanning set for L in $\langle 2S \rangle$. Suppose also that $\dot{\mathcal{M}}$ is a coset spanning set for \dot{R}_{lg} in $2\langle \dot{R}_{sh} \rangle$. For each $j \in J$ and $k \in K$, pick $\dot{\alpha}_j \in \dot{R}_{sh}$ and $\dot{\beta}_k \in \dot{R}_{lg}$ and set $\Pi := \Pi_{sh} \cup \Pi_{lg}$ where $\Pi_{sh} := \{\dot{\alpha}_j + \sigma_j \mid j \in J\}$ and $\Pi_{lg} := \dot{\mathcal{M}} \cup \{\dot{\beta}_k + \tau_k \mid k \in K\}$. Further, suppose that

$$\langle \mathcal{R} \rangle + \langle \mathcal{M} \rangle = \langle S \rangle. \quad (3.17)$$

Then Π is a reflectable set for R .

Proof. First we note that since \mathcal{M} is a coset spanning set for L in $2\langle S \rangle$, we have $\langle \mathcal{M} \rangle + 2\langle S \rangle = \langle L \rangle + 2\langle S \rangle = L$, therefore by (3.17),

$$\langle \mathcal{M} \rangle + 2\langle \mathcal{R} \rangle = L. \quad (3.18)$$

Since $0 \in \mathcal{R}$, Π contains at least a short root $\dot{\alpha} \in \dot{R}$. Then we have from Proposition 2.4(iii) that $\{\dot{\alpha}\} \cup \dot{\mathcal{M}}$ is a reflectable set for the locally finite root system \dot{R} . So

$$\dot{R}^\times \subseteq \mathcal{W}_\Pi \Pi \quad \text{and} \quad \dot{\mathcal{W}} \subseteq \mathcal{W}_\Pi. \quad (3.19)$$

From this, and the fact that $\alpha \in \mathcal{W}_\Pi \Pi$ if and only if $-\alpha \in \mathcal{W}_\Pi \Pi$, we get

$$\dot{R}_{sh} \pm \sigma_j = \dot{\mathcal{W}}(\dot{\alpha}_j \pm \sigma_j) \subseteq \mathcal{W}_\Pi \Pi; \quad j \in J, \quad (3.20)$$

and

$$\dot{R}_{lg} \pm \tau_k = \dot{\mathcal{W}}(\dot{\beta}_k \pm \tau_k) \subseteq \mathcal{W}_\Pi \Pi; \quad k \in K. \quad (3.21)$$

Then for $i, j \in J$ and $k \in K$,

$$\dot{R}_{sh} \pm \sigma_j \pm \tau_k \pm 2\sigma_i \subseteq \dot{\mathcal{W}} \mathcal{W}_{\dot{R}_{sh} \pm \sigma_i} \mathcal{W}_{\dot{R}_{lg} \pm \tau_k} (\dot{R}_{sh} \pm \sigma_j) \subseteq \mathcal{W}_\Pi \Pi.$$

It follows, by repeating this argument, that for a fixed $j \in J$,

$$\dot{R}_{sh} + \sigma_j + \langle \mathcal{M} \rangle + 2\langle \mathcal{R} \rangle = \dot{R}_{sh} + \sigma_j + \langle \tau_k \mid k \in K \rangle + 2\langle \sigma_i \mid i \in J \rangle \subseteq \mathcal{W}_{\Pi\Pi}.$$

Thus by (3.18), $\dot{R}_{sh} + \sigma_j + L \subseteq \mathcal{W}_{\Pi\Pi}$ for all $j \in J$. Now since $\cup_{j \in J}(\sigma_j + L) = S$, we get

$$\dot{R}_{sh} + S \subseteq \mathcal{W}_{\Pi\Pi}. \quad (3.22)$$

Next, we show that $\dot{R}_{lg} + L \subseteq \mathcal{W}_{\Pi\Pi}$. Since $|I| \geq 3$, there exist $\dot{\beta}_1, \dot{\beta}_2 \in \dot{R}_{lg}$ such that $(\dot{\beta}_1, \dot{\beta}_2^\vee) = \pm 1$. This together with (3.19)-(3.21) shows that for $j \in J$ and $k, t \in K$, we have

$$\dot{R}_{lg} \pm \tau_k \pm 2\sigma_j \pm \tau_t \subseteq \dot{\mathcal{W}}_{\dot{R}_{lg} \pm \tau_t} \mathcal{W}_{\dot{R}_{sh} \pm \sigma_j}(\dot{R}_{lg} \pm \tau_k) \subseteq \mathcal{W}_{\Pi\Pi}.$$

By repeating this argument we get

$$\dot{R}_{lg} + \langle \mathcal{M} \rangle + 2\langle \mathcal{R} \rangle = \dot{R}_{lg} + \langle \tau_k \mid k \in K \rangle + 2\langle \sigma_j \mid j \in J \rangle \subseteq \mathcal{W}_{\Pi\Pi},$$

and so by (3.18)

$$\dot{R}_{lg} + L \subseteq \mathcal{W}_{\Pi\Pi}. \quad (3.23)$$

Now (3.22) and (3.23) show that Π is a reflectable set for R . \square

By a B_I -reflectable data, we mean a tuple

$$(\dot{R}, S, L, \dot{\mathcal{M}}, \{\dot{\alpha}_i\}_{i \in I}, \{\dot{\beta}_k\}_{k \in K}, \mathcal{R}, \mathcal{M})$$

in which the ingredients satisfy the conditions in the statement of Lemma 3.15.

Theorem 3.24. *A subset Π of R^\times with $\langle \Pi \rangle = \langle R \rangle$ is a reflectable set (resp. a reflectable base) for R if and only if $\Pi_{sh} = R_{sh} \cap \Pi$ is a strong coset spanning set (a minimal strong coset spanning set) for R_{sh} in $\langle R_{lg} \rangle$ and $\Pi_{lg} = R_{lg} \cap \Pi$ is a coset spanning set (a minimal coset spanning set) for R_{lg} in $2\langle R_{sh} \rangle$.*

Proof. By Lemma 3.5, it is enough to prove that any subset Π of R^\times with $\langle \Pi \rangle = \langle R \rangle$ such that Π_{sh} is a strong coset spanning set for R_{sh} in $\langle R_{lg} \rangle$ and Π_{lg} is a coset spanning set for R_{lg} in $2\langle R_{sh} \rangle$, is a reflectable set for R . So take Π to be such a subset of R^\times . Then $\Pi = \mathcal{R}_0 \cup \mathcal{M}_0$, where \mathcal{R}_0 is a strong coset spanning set for R_{sh} in $\langle R_{lg} \rangle$ and \mathcal{M}_0 is a coset spanning set for R_{lg} in $2\langle R_{sh} \rangle$. It follows that $\bar{\Pi} = \bar{\mathcal{R}}_0 \cup \bar{\mathcal{M}}_0$ and that $\bar{\mathcal{R}}_0$ is a nonempty subset of \bar{R}_{sh} and $\bar{\mathcal{M}}_0$ is a coset spanning set for \bar{R}_{lg} in $2\langle \bar{R}_{sh} \rangle$. Let $\bar{\alpha} \in \bar{\mathcal{R}}_0$. Then by Proposition 2.4, $\{\bar{\alpha}\} \cup \bar{\mathcal{M}}_0$ is a reflectable set for \bar{R} , so is $\bar{\Pi}$. Thus by Corollary 2.14 and Proposition 2.14, it contains an integral reflectable base \mathcal{P} . Take $\dot{\Pi} \subseteq \Pi$ to be a pre-image of \mathcal{P} and construct \dot{R} and (S, L) as usual to get a description of R in the form (3.16), then $\dot{\Pi}$ is a reflectable base for \dot{R} and so by Proposition 2.4, there is a short root $\dot{\alpha}$ and a coset basis $\dot{\mathcal{M}} \subseteq \mathcal{M}_0$ for \dot{R}_{lg} in $\langle \dot{R}_{sh} \rangle$ such that $\dot{\Pi} = \{\dot{\alpha}\} \cup \dot{\mathcal{M}}$.

Next fix $\dot{\beta} \in \dot{R}_{lg}$ and take $\delta \in L$, then $\dot{\beta} + \delta \in \langle R_{lg} \rangle = \langle \mathcal{M}_0 \rangle + 2\langle R_{sh} \rangle$ and so $\delta \in \mathfrak{p}(\langle \mathcal{M}_0 \rangle + 2\langle R_{lg} \rangle) = \langle \mathfrak{p}(\mathcal{M}_0 \setminus \dot{\mathcal{M}}) \rangle + \langle 2S \rangle$. Thus $L = \langle \mathfrak{p}(\mathcal{M}_0 \setminus \dot{\mathcal{M}}) \rangle + \langle 2S \rangle$, which implies that $\mathfrak{p}(\mathcal{M}_0 \setminus \dot{\mathcal{M}})$ is a coset spanning set for L in $\langle 2S \rangle$.

Next, assume $\mathcal{R}_0 = \{\alpha_j \mid j \in J\}$, $\mathcal{M}_0 \setminus \dot{\mathcal{M}} = \{\beta_k \mid k \in K\}$ and set

$$\mathcal{R} := \mathfrak{p}(\mathcal{R}_0) = \{\sigma_j := \mathfrak{p}(\alpha_j) \mid j \in J\} \quad \text{and} \quad \mathcal{M} := \mathfrak{p}(\mathcal{M}_0 \setminus \dot{\mathcal{M}}) = \{\tau_k := \mathfrak{p}(\beta_k) \mid k \in K\}.$$

Note that $0 \in \mathcal{R} \subseteq S$, as $\dot{\alpha} \in \mathcal{R}_0 \cap \dot{R}$. Now as in the case of B_2 , one sees that \mathcal{R} is a strong coset spanning set for S in L . It also follows that \mathcal{M} is a coset spanning set for L in $2\langle S \rangle$. Therefore, up to this point we have seen that the tuple

$$(\dot{R}, S, L, \dot{\mathcal{M}}, \{\dot{\alpha}_i\}_{i \in I}, \{\dot{\beta}_k\}_{k \in K}, \mathcal{R}, \mathcal{M})$$

is a B_I reflectable-data, provided that (3.17) holds. But we know that $\langle \Pi \rangle = \langle R \rangle$, so $\langle \mathcal{R}_0 \cup \mathcal{M}_0 \rangle = \langle \dot{R} \rangle + \langle S \rangle$. Thus

$$\langle S \rangle = \mathfrak{p}(\langle \mathcal{R}_0 \rangle + \langle \mathcal{M}_0 \rangle) = \langle \mathfrak{p}(\mathcal{R}_0) \rangle + \langle \mathfrak{p}(\mathcal{M}_0 \setminus \dot{\mathcal{M}}) \rangle = \langle \mathcal{R} \rangle + \langle \mathcal{M} \rangle,$$

so (3.17) holds. Now by Lemma 3.15, Π is a reflectable set. The assertion in the statement now follows immediately from Lemma 3.5. \square

Types $\dot{A}_I(|I| \geq 3)$ $D_I(|I| \geq 4)$: For the types under consideration, we have

$$R^\times = \dot{R}^\times + S, \quad (3.25)$$

where S is a pointed reflection subspace of A^0 with $\langle S \rangle = S$ and \dot{R} is a locally finite root system of the corresponding type.

Theorem 3.26. $\Pi \subseteq R^\times$ is a reflectable set (resp. reflectable base) for R if and only if Π is a generating set (resp. minimal generating set) for $\langle R \rangle$.

Proof. We know that any reflectable set for R is a generating set for $\langle R \rangle$, so we assume $\Pi \subseteq R^\times$ is a generating set for $\langle R \rangle$ and show that it is a reflectable set for R . By definition $\langle \Pi \rangle = \langle R \rangle$, and so $\langle \bar{\Pi} \rangle = \langle \bar{R} \rangle$. Now by Proposition 2.5, $\bar{\Pi}$ is a reflectable set for \bar{R} , so by Propositions 2.6 and 2.9, it contains an integral reflectable base \mathcal{P} . Let $\dot{\Pi} \subseteq \Pi$ be a pre-image of \mathcal{P} and construct \dot{R} and S in the usual way to get a description of R in the form (3.25). Now $\dot{\Pi}$ is a reflectable base for \dot{R} and so $\langle \dot{\Pi} \rangle = \langle \dot{R} \rangle$. Set $\mathcal{M} = \Pi \setminus \dot{\Pi}$. Since $\langle \dot{R} \rangle + S = \langle R \rangle = \langle \Pi \rangle$, we have $S = \langle \mathfrak{p}(\Pi) \rangle = \langle \mathfrak{p}(\Pi \setminus \dot{\Pi}) \rangle = \langle \mathfrak{p}(\mathcal{M}) \rangle$. Let $\mathcal{M} = \{\alpha_j \mid j \in J\}$ and for $j \in J$ set $\sigma_j := \mathfrak{p}(\alpha_j)$ and $\dot{\alpha}_j = \alpha_j - \sigma_j$. Since $\dot{\Pi} \subseteq \Pi$, we have

$$\dot{R}^\times = \mathcal{W}_{\dot{\Pi}} \dot{\Pi} \subseteq \mathcal{W}_{\Pi} \Pi \quad \text{and} \quad \dot{\mathcal{W}} \subseteq \mathcal{W}_{\Pi}.$$

Now for any $j \in J$, we have $\dot{R}^\times + \sigma_j = \dot{\mathcal{W}}(\dot{\alpha}_j + \sigma_j) \subseteq \mathcal{W}_{\Pi} \Pi$. Since $\alpha \in \mathcal{W}_{\Pi} \Pi$ if and only if $-\alpha \in \mathcal{W}_{\Pi} \Pi$, we have $\dot{R}^\times \pm \sigma_j \subseteq \mathcal{W}_{\Pi} \Pi$. It is known that for the types under consideration for any $\dot{\alpha} \in \dot{R}^\times$, there exists $\dot{\beta} \in \dot{R}^\times$ with $(\dot{\alpha}, \dot{\beta}^\vee) = \pm 1$. Therefore for any $i, j \in J$, we have

$$\dot{R}^\times \pm \sigma_j \pm \sigma_i \subseteq \dot{\mathcal{W}} \mathcal{W}_{\dot{R}^\times \pm \sigma_j}(\dot{R}^\times \pm \sigma_i) \subseteq \mathcal{W}_{\Pi} \Pi.$$

Repeating this argument, we get $\dot{R}^\times + \langle \sigma_j \mid j \in J \rangle \subseteq \mathcal{W}_{\Pi} \Pi$, so

$$R^\times = \dot{R}^\times + S = \dot{R}^\times + \langle \mathfrak{p}(\mathcal{M}) \rangle = \dot{R}^\times + \langle \sigma_j \mid j \in J \rangle \subseteq \mathcal{W}_{\Pi} \Pi.$$

This shows that Π is a reflectable set and completes the proof. \square

Types F_4, G_2 :

Theorem 3.27. *Suppose that \mathcal{P} is a subset of R^\times with $\langle \mathcal{P} \rangle = \langle R \rangle$. Then \mathcal{P} is a reflectable set (resp. reflectable base) for R if and only if $\mathcal{P}_{sh} = \mathcal{P} \cap R_{sh}$ is a coset spanning set (resp. minimal coset spanning set) for R_{sh} in $\langle R_{lg} \rangle$ and $\mathcal{P}_{lg} = \mathcal{P} \cap R_{lg}$ is a coset spanning set (resp. minimal coset spanning set) for R_{lg} in $\rho \langle R_{sh} \rangle$.*

Proof. Consider the description

$$R^\times = (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \quad (3.28)$$

for R where S and L are as in Proposition 1.13 with $\rho = 2$ for type F_4 and $\rho = 3$ for type G_2 . We know that $\langle S \rangle = S$ and $\langle L \rangle = L$. let $\mathcal{P} \subseteq R^\times$ be a reflectable set for R . Then by Lemma 1.22,

$$R_{sh} = \mathcal{W}_{\mathcal{P}} \mathcal{P}_{sh} = \bigcup_{\alpha \in \mathcal{P}_{sh}} \mathcal{W}_{\mathcal{P}}(\alpha) \subseteq \langle \mathcal{P}_{sh} \rangle + \langle R_{lg} \rangle$$

and

$$R_{lg} = \mathcal{W}_{\mathcal{P}} \mathcal{P}_{lg} = \bigcup_{\alpha \in \mathcal{P}_{lg}} \mathcal{W}_{\mathcal{P}}(\alpha) \subseteq \langle \mathcal{P}_{lg} \rangle + \langle \rho R_{sh} \rangle.$$

Thus we have $\langle R_{sh} \rangle = \langle \mathcal{P}_{sh} \rangle + \langle R_{lg} \rangle$ and $\langle R_{lg} \rangle = \langle \mathcal{P}_{lg} \rangle + \langle \rho R_{sh} \rangle$. So by definition \mathcal{P}_{sh} is a coset spanning set for R_{sh} in $\langle R_{lg} \rangle$ and \mathcal{P}_{lg} is a coset spanning set for R_{lg} in $\langle \rho R_{sh} \rangle$. Now as \mathcal{P} is a generating set for $\langle R \rangle$, we are done.

Next Suppose that $\Pi = \mathcal{M}_1 \cup \mathcal{M}_2$, where \mathcal{M}_1 is a coset spanning set for R_{sh} in $\langle R_{lg} \rangle$ and \mathcal{M}_2 is a coset spanning set for R_{lg} in $\langle \rho R_{sh} \rangle$. Then $\bar{\Pi} = \bar{\mathcal{M}}_1 \cup \bar{\mathcal{M}}_2$ such that $\bar{\mathcal{M}}_1$ is a coset spanning set for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$ and $\bar{\mathcal{M}}_2$ is a coset spanning set for \dot{R}_{lg} in $\langle \rho \dot{R}_{sh} \rangle$. Thus by Propositions 2.12 and 2.10, $\bar{\Pi}$ is a reflectable set for \bar{R} and so by Corollary 2.14, it contains an integral reflectable base \mathcal{P} , for which we fix a pre-image $\dot{\Pi} \subseteq \Pi$, and we construct \dot{R} , S and L in the usual way to get a description of R in the form (3.28). Then $\dot{\Pi}$ is a reflectable base for \dot{R} . By Propositions 2.12 and 2.10, $\dot{\Pi} = \dot{\mathcal{M}}_1 \cup \dot{\mathcal{M}}_2$ where $\dot{\mathcal{M}}_1$ is a coset basis for \dot{R}_{sh} in $\langle \dot{R}_{lg} \rangle$ and $\dot{\mathcal{M}}_2$ is a coset basis for \dot{R}_{lg} in $\langle \rho \dot{R}_{sh} \rangle$. Let $\mathcal{M}_1 \setminus \dot{\mathcal{M}}_1 = \{\alpha_j \mid j \in J\}$ and $\mathcal{M}_2 \setminus \dot{\mathcal{M}}_2 = \{\beta_k \mid k \in K\}$. For each $j \in J$, we have $\alpha_j = \dot{\alpha}_j + \sigma_j$ where $\sigma_j = \mathfrak{p}(\alpha_j)$ and $\dot{\alpha}_j = \alpha_j - \mathfrak{p}(\alpha_j)$. Similarly, for each $k \in K$, $\beta_k = \dot{\beta}_k + \tau_k$ where $\tau_k = \mathfrak{p}(\beta_k)$ and $\dot{\beta}_k = \beta_k - \mathfrak{p}(\beta_k)$. Take $\mathcal{M}'_1 = \mathfrak{p}(\mathcal{M}_1 \setminus \dot{\mathcal{M}}_1) = \{\sigma_j \mid j \in J\}$ and $\mathcal{M}'_2 = \mathfrak{p}(\mathcal{M}_2 \setminus \dot{\mathcal{M}}_2) = \{\tau_k \mid k \in K\}$. One can check that \mathcal{M}'_1 is a coset spanning set for S in L and \mathcal{M}'_2 is a coset spanning set for L in ρS . Since $\langle \mathcal{P} \rangle = \langle R \rangle$, it follows that

$$\langle \mathcal{M}'_1 \rangle + \langle \mathcal{M}'_2 \rangle = \langle S \rangle \quad \text{and} \quad 2\langle \mathcal{M}'_1 \rangle + \langle \mathcal{M}'_2 \rangle = L. \quad (3.29)$$

Now as $\dot{\Pi} \subseteq \Pi$, we have

$$\dot{R}^\times \subseteq \mathcal{W}_{\Pi} \Pi \quad \text{and} \quad \dot{\mathcal{W}} \subseteq \mathcal{W}_{\Pi}.$$

From this it follows that

$$\dot{R}_{sh} \pm \sigma_j \subseteq \dot{\mathcal{W}}(\dot{\alpha}_j \pm \sigma_j) \subseteq \mathcal{W}_{\Pi} \Pi \quad \text{and} \quad \mathcal{W}_{\dot{R}_{sh} \pm \sigma_j} \subseteq \mathcal{W}_{\Pi}, \quad (j \in J)$$

and

$$\dot{R}_{lg} \pm \tau_k \subseteq \dot{\mathcal{W}}(\dot{\beta}_j \pm \tau_k) \subseteq \mathcal{W}_{\Pi} \Pi \quad \text{and} \quad \mathcal{W}_{\dot{R}_{lg} \pm \tau_k} \subseteq \mathcal{W}_{\Pi}, \quad (k \in K).$$

Next considering some basic facts on finite root systems of types F_4 and G_2 , we have for $i, j \in J$ and $t, k \in K$,

$$\dot{R}_{sh} \pm \sigma_j \pm \sigma_i \pm \tau_k \pm \tau_t \subseteq \dot{W}W_{\dot{R}_{sh} \pm \sigma_i} W_{\dot{R}_{lg} \pm \tau_t} W_{\dot{R}_{lg} \pm \tau_k} (\dot{R}_{sh} \pm \sigma_j) \subseteq W_{\Pi} \Pi.$$

Repeating this argument, and using (3.29), we obtain

$$\dot{R}_{sh} + S = \dot{R}_{sh} + \langle \mathcal{M}'_1 \rangle + \langle \mathcal{M}'_2 \rangle \subseteq W_{\Pi} \Pi.$$

We also have for $i, j \in J$ and $t, k \in K$,

$$\dot{R}_{lg} \pm \tau_k \pm \tau_t \pm \sigma_i \pm \sigma_j \subseteq \dot{W}W_{\dot{R}_{lg} \pm \tau_t} W_{\dot{R}_{sh} \pm \sigma_i} W_{\dot{R}_{sh} \pm \sigma_j} (\dot{R}_{lg} \pm \tau_k) \subseteq W_{\Pi} \Pi.$$

Again, repeating this argument and using (3.29), we get

$$\dot{R}_{lg} + L = \dot{R}_{lg} + 2\langle \mathcal{M}'_1 \rangle + \langle \mathcal{M}'_2 \rangle \subseteq W_{\Pi} \Pi.$$

These all together show that Π is a reflectable set. This completes the proof. \square

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